POLSKA AKADEMIA NAUK
INSTYTUT FIZYKI

ACTA PHYSICA POLONICA

DWUMIESIĘCZNIK

Vol. XV — Fasc. 3

WARSZAWA 1956

First reprint by PWN — Polish Scientific Publishers 1969

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PAŃSTWOWE WYDAWNICTWO NAUKOWE

IMPURITY-ACTIVATED CRYSTALLINE PHOSPHORS THEIR PRODUCTION AND THERMOLUMINESCENCE CURVES

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(*Received November 22, 1955*)

Some useful observations made when searching for methods of making strongly glowing Phosphors (of not very pure materials) are reported. A simple apparatus has been constructed which traces the glow curve directly on the photographic paper by means of a light spot. A uniform increase of temperature is easy to obtain. Some observations concerning the influence of Mg on the depth electron traps have been made. Two new methods of finding the depth of electron traps (without approximate calculations) have been applied to the produced phosphors.

I. Introduction

This paper concerns those crystalline phosphors for which a necessary condition of glowing is an admixture of impurities, called activators. The activators are usually some heavy metals. Natural impurity-activated crystalline phosphors have been known ever since the 17-th century, while methods of their artificials production have been studied thoroughly since the 19-th century. In this respect the greatest credit goes to Lenard and his school. Later on several investigators worked out different methods for producing these phosphors. This problem is still of interest on account of the great variety of phosphors and their increasing practical importance. The investigation of their phosphorescence and photoconductivity is also important for the understanding of their structure and the mechanism of photoluminescence. In particular, the investigation of glow curves enables the depth of electron traps to be determined, i. e., the difference of energy between the localized level and the bottom of the conductivity band. The mean time interval during which electrons remain on these levels determines the mean duraction of phosphorescence and in connected with the depth of electron traps by

$$\frac{1}{\tau} = p_0 e^{\frac{\varepsilon}{KT}} \tag{1}$$

where τ is the mean life-time, p_0 is a constant which is characteristic for a given phosphor, and ε is the depth for the trap. Numerous investigations have shown that

some of these traps are activator-independent traps and are connected with natural defects of the cristalline structure, while the remaining activator-dependent traps are due to the activator, or to the so called co-activator.

II. Production of impurity-activated cristalline phosphors

The first part of this paper is devoted to methods of producing cristalline phosphors with long life-times. Such phosphors are best suited for investigations of glow curves. The phosphors produced from: CaS, SrS, and BaS have particularly lasting phosphorescence. The methods of their production are well known and described in the literature, but the utmost purity of materials is necessary for their production. In some cases even a purity above the "spectrally pure" is needed, otherwise the intensity of phosphorescence is considerably reduced. Such substances are not accessible and suitable quartz and platinum apparatus for purification was not available. Neverthelless, we attempted to produce phosphors from the available materials viz. from $CaCO_3$, $SrCO_3$, and $BaCO_3$, sulphur, several salts of light metals for melting, as well as the metals Bi, Cu, Tl, Mn, Zn, Pb as activators.

As a rule we followed the methods to be found in the literature especially those of Lenard and Tomaschek; but searching for strongly luminescent phosphors, derivable from the available substances, we noticed some useful facts:

(i) Some of the melting materials recommended in the literature may be replaced successfully by some others. For example, when producing the BaS-Cu phosphors, we used $Na_2B_4O_7$ for the melting instead of the recommended Li_3PO_4, and obtained a more strongly glowing phosphor.

(ii) While making several trials, we found that the way in which the firing of the phosphor is carried out is of importance for the strength of luminescence. In particular, if the substance to be heated is put into the oven at a comparatively low temperature (e. g. CaS—Bi at 400°C) so that the increase of temperature up to about 1100°C is slow, we get a weakly phosphorescent-low quality phosphor.

(iii) An admixture of $MgCO_3$ and an admixture of glucose to the mixture to be heated usually increases the intensity of luminescence.

For heating of phosphors, quartz crucibles, or, more frequently, crucibles of glaze porcelain were used together with a suitable layer, to prevent contamination of the phosphor by the glaze. After several trials, we obtained the following phosphors which are distinguished by intense luminescence.

1) CaS—Bi luminescence violet

2) The mixture CaS—BaS—SrS—Bi, green or blue according to the proportion of the three basic substances.

3) BaS—Cu orange

4) CaS—Tl green, weaker than the above mentioned phosphors.

These phosphors glow strongly when excited by the light of a mercury lamp, but phosphors 1 and 2 are also good if excited by daylight.

III. Glow curves

A. The second part of this paper concerns the investigation of the glow curves of the produced phosphors. An estimate of the depth of electron traps is of theoretical value, since it may elucidate the character of the structure defects causing the appearance of those levels.

Moreover it is of practical value for applications involving phosphors to know the position of the glow-curve (thermoluminescence curve) maximum.

The glow curve is the plot of intensity of luminescence versus temperature, where the temperature is increasing linearly in time. In order to obtain entire glow curve it is necessary to excite the phosphor at a sufficiently low temperature, so that the phosphorescence is completely frozen-in. Then, after interrupting the excitation, one measures the change of intensity by increasing the temperature linearly in time.

B. Apparatus. Our apparatus was constructed so as to enable the temperature to be increased continuously and uniformly, and to allow simultaneous measurements of temperature and relative intensity of luminescence. The arrangement for changing temperature was similar to that used by Randal et al. (1945). It consists of copper box with a channel for liquid air surrounded by an electric heater coil. Both are thouroughly isolated from the outer walls of the box, except for one of the vertical walls, which is in contact with liquid air and with the heater coil. The phosphor was attached to this wall (Fig. 1).

Several ways of fastening were tried (e. g. by means of water glass or polystyrene in benzene. No significant influence of the glue was noticed. Usually however, the phosphor was held on the wall without glue, with the help of a quartz window and hard strings. One of the thermocouple junctions was fastened to the same wall. It should be placed so as to indicate the actual temperature of the phosphor without making it difficult to press the phosphor to the wall. The junction was flat and situated in a shallow cavity in the middle of the wall. The current produced by thermocouple passed through a properly shunted galvanometer. The shunt was adjusted so that the change of temperature $\Delta T = 1°$ corresponded to a displacement of the light spot (temperature spot) by 1 mm on the scale. The window pressing against the phosphor froze during excitation. It is evident, that the most practical means to prevent this is to use a double window in form of a cylinder 25 mm long with a cavity into which small quantity of P_2O_5 was placed to remove the moisture.

The apparatus for the measuring the relative intensity of phosphorescence consisted of a photocell, a d.c. amplifier and a mirror galvanometer. The phosphor was cooled up to about 130°K and then excited with a mercury arc. After switching off the lamp, the phosphor was turned towards the photocell by a special arrangement. The shutter of the metallic shield to the photocell was opened for 5 sec. after the excitation was switched off. The amplified current was detected by the mirror galvanometer. The light spot from this galvanometer was reflected on to a screen placed at a distance of 1 m

and indicated the relative intensity of phosphorescence. The remaining part of the apparatus was constructed so that the glow curve was traced by the light spot directly on the photographic paper moving uniformly with the screen in a vertical direction. At the same time, the temperature spot traces the change of temperature in time on the same paper. The photographic paper was placed under a transparent scale. In order to obtain a true glow curve, the temperature of the phosphor must increase linearly with time. In order to secure this, a curve (corresponding to the curve gauging the thermocouple) was drawn on the above mentioned transparent scale. In order to obtain a uniformly rising temperature, it was necessary to adjust the heating current, thus constraining the temperature spot to move along the curve drawn on the scale. Obviously, this curve has to be adjusted to the desired speed of heating, whereas the speed of the screen is fixed. The speeds of heating used were usually 1 grad/sec. and 0.5 grad/sec., while the speed of the screen was 0.5 mm/sec. The temperature range for which the glow curves were obtained was 130°–600°K. After the two spots have traced the intensity and temperature curves, the entire screen was exposed to the light for a few seconds, in order to obtain the negatives of the scale and the temperature pattern curve as well.

The rate of temperature rise was regulated by means of a resistance in the heating circuit. The role of the experimenter is to observe the temperature spot and to regulate

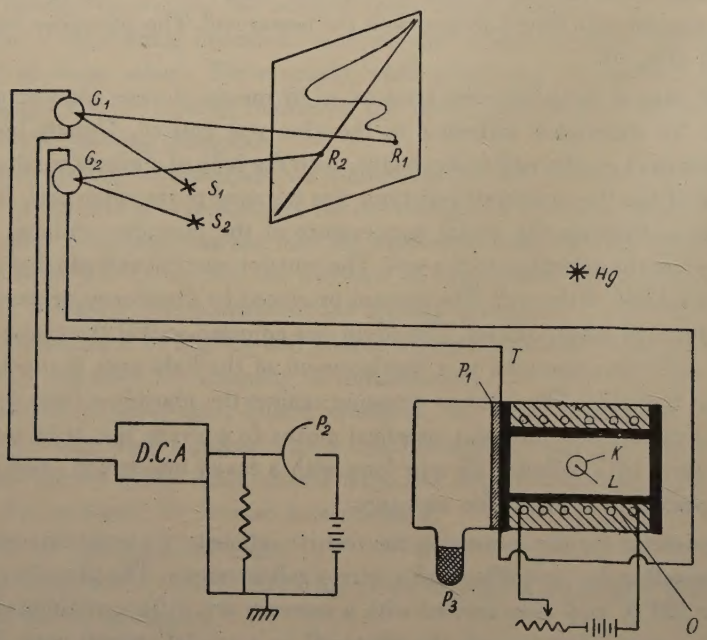

Fig. 1. Block diagram of the apparatus. *Hg* — mercury lamp, *O* — electric heater coil, *T* — thermocouple, P_1 — investigated phosphor, P_2 — photocell, P_3 — phosphorus-pentoxide, *D. C. A.* — *D. C.* Ampifier, G_1, G_2 — mirror galvanometers, R_1 — intensity light spot, R_2 — thermocouple light spot, S_1, S_2 — bulbs. *K* — opening of the channel for liquid air, *L* — axis of rotation.

the resistance so that the spot moves along the pattern curve. After a certain amount of training this could be achieved quite satisfactorily, as may be seen on the curves of Fig. 2 and Fig. 3. The whole apparatus was operated by a single person.

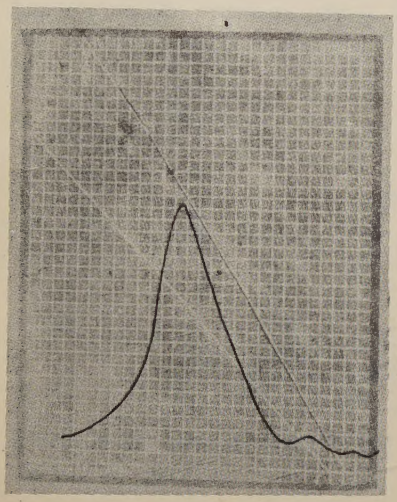

Fig. 2. Example of a glow curve for CaS-Bi (with Mg) temperature range 180°K—570°K. Original.

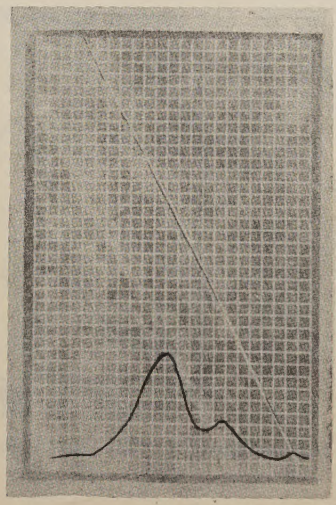

Fig. 3. Example of a glow curve for green phosphor. Temperature range 135°K—570°K. Original.

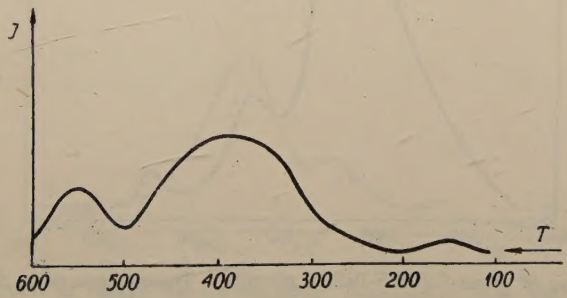

Fig. 4. Glow curve of violet CaS-Bi phosphor (without Mg). Temperature range 130°K—600°K.

C. Discussion of glow curves. The glow curves of two kinds of CaS—Bi phosphors have been investigated. One of them was produced by heating at a temperature of 1100°C the mixture: 6g CaCO + lg S + 0.2g glucose + 0.2g Na_2SO_4 + 0.2g K_2SO_4 + 0.0002g Bi, the other was produced by heating under the same conditions a mixture which contained additionally lg MgO. It appeared that the mixture containing this last substance increases the brightness of the phosphor and causes considerable changes in the distribution of the trap depths. The glow curve of the so called "normal" CaS—Bi phosphor may be found in the literature (Garlick 1945). It appears that it is precisely the thermoluminescence curve of our phosphor obtained

without the mixture of MgO. This curve is given in Fig. 4 for comparison with the thermoluminescence curve for the same phosphor, with the mixture of MgO (Fig. 5). The curve (4) like the curve in Garlick's book for the "normal" CaS—Bi phosphor possesses three maxima. The first of them, a very weak one, corresponds to the temperature 153°K. The second about 390°—400°K is the strongest maximum and is

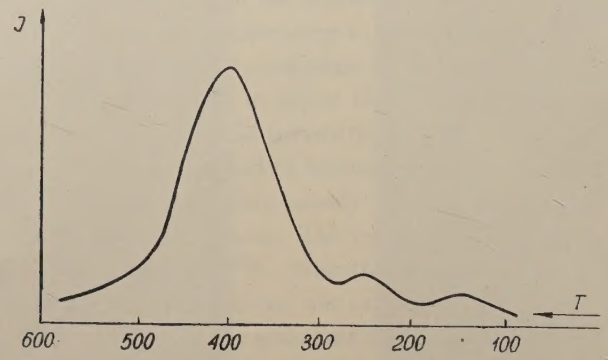

Fig. 5. Glow curve of violet CaS-Bi phosphor (with Mg). Temperature range 130°K—600°K.

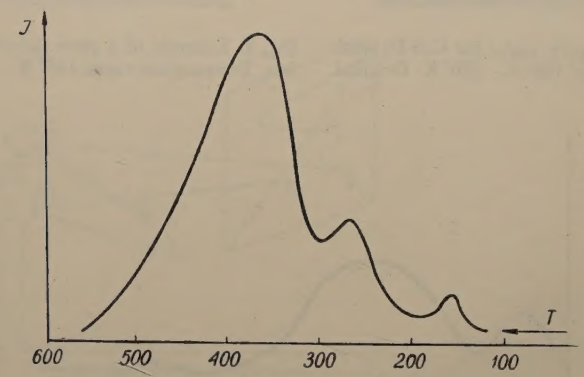

Fig. 6. Glow curve of green CaS-SrS-BaS-Bi phosphor. Temperature range 130°K—600°K.

rather diffuse. The third one, about 553°K, is nearly as strong as the second. Fig. 5 represents the glow curve of the CaS—Bi phosphor with admixture of Mg differing widely from the previous case. Besides a weak maximum at about 153°K there exists also a new weak maximum at about 258°K. The third maximum about 390°—406°K is much stronger than the corresponding maximum of Fig. 4. On the other hand, the maximum at the high temperature (of 553°K) is missing. It is seen that the admixture of MgO considerably influences the structure of the electrons traps, i.e., their depth (appearance of new maxima and disappearance of other maxima) as well as the concentration of traps of a given depth (the increase of the intensity at about 400°K).

Since the maxima at about 255°—270° K appear also for other phosphors with

an admixture of Mg (Fig. 6 and Fig. 7) it seems that Mg plays an essential role in the forming of traps corresponding to this maximum.

The glow of green and blue phosphors are very similar and possess three maxima. The main difference is that in the former, the maximum appearing at the highest

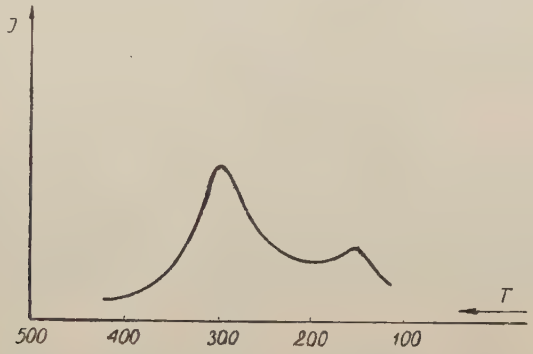

Fig. 7. Glow curve of blue CaS-SrS-BaS-Bi phosphors.

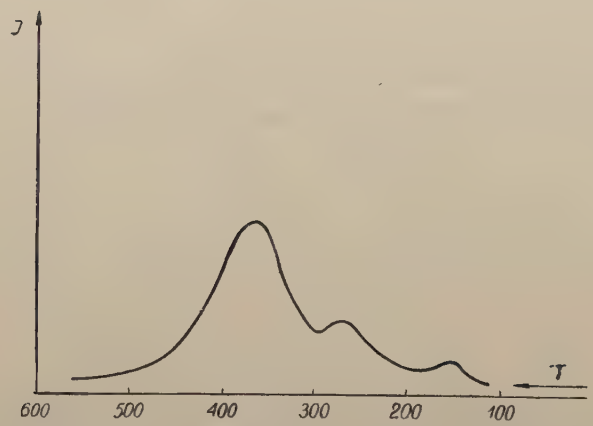

Fig. 8. Glow curve of orange BaS-Cu phosphor. Temperature range 100°K—500°K.

temperature is shifted by about +10°K. Perhaps it is connected with the fact that the green phosphor contains more CaS. This supposition is plausible, since the violet phosphor BaS—Bi (whose basic substance is CaS) possesses a maximum at a still higher temperature.

The BaS—Cu phosphor orange, possesses no high temperature peak. Therefore its life-time at normal temperature is shorter than the case of the above mentioned phosphors. Its maxima correspond to the temperatures 150°K and 296°K (Fig. 8).

D. The depth of electron traps. The depths of traps have been determined from the temperatures of the maxima of the thermoluminescence curves. To this end was found the extremum of the theoretical curve derived under the assumption that

the retrapping effects (Randall at al. 1945) does not exist. Although the neglect of retrapping does not seem to be well founded (at least in the case of photo-conductive phosphors) nevertheless, it appears to have little influence upon the temperatures at which the peaks of glow curves appear. The error caused by the neglect of retrapping in the computation of ε does not exceed 3 per cent i. e., it lies within the limits of experimental error. Under the assumption that all traps have the same depth the expression for the glow curve without retrapping

$$I = C\, n_0\, e^{-\int_{T_0}^{T} \frac{p_0}{\beta} e^{-\frac{\varepsilon}{KT}} dT} \cdot p_0\, e^{-\frac{\varepsilon}{KT}} \tag{2}$$

where C fixes the unit fot I and may be taken $C = 1$, T_0 is the temperature of excitation, n_0 is the density of the excited centres at the moment the excitation is switched off, β – the speed of heating phosphor in grad/sec, T – the temperature at a given instant. The condition for the maximum of the curve (2) yields

$$\left(\frac{-p_0}{\beta} \cdot e^{-\frac{\varepsilon}{KT_m}} + \frac{\varepsilon}{KT_m^2}\right) n_0\, e^{-\frac{p_0}{\beta}\int_{T_0}^{T_m} e^{-\frac{\varepsilon}{KT}} dT} \cdot p_0 \cdot e^{-\frac{\varepsilon}{KT_m}} = 0$$

from which

$$\frac{\varepsilon}{KT_m} = \frac{p_0}{\beta}\, e^{-\frac{\varepsilon}{KT_m}}$$

where T_m is the temperature of maximum of the curve. Thus,

$$e^{-\frac{\varepsilon}{KT_m}} = \frac{\beta\, \varepsilon}{p_0\, KT_m^2} \tag{3}$$

From (3), ε may be determined by a graphical method. Denoting

$$z = \frac{\varepsilon}{KT_m}\,; \quad U = \frac{p_0\, T_m}{\beta}$$

we have

$$z \cdot e^z = U, \text{ or } z + ln\, z = ln\, U$$

From the curve $z + ln\, z$ versus z, we find z and ε for a given U. In order to know U, we must know p_0 for a given phosphor. According to Randall, p_0 is of the order $10^{8\pm1}$ sek^{-1} for phosphors whose basic materials are alkaline earth sulfide. In this paper the value $p_0 = 10^8$ sek^{-1} has been assumed. We have also applied another method for computing the depth of traps. In this method, it is necessary to know only two temperatures, T_1 and T_2 of the same maximum corresponding to two different speeds of heating β_1 and β_2. Since the depth is independent of the heating speed, we find from (3)

$$T_1\, ln \frac{p_0 K}{\varepsilon} \cdot \frac{T_1^2}{\beta_1} = T_2\, ln \frac{p_0 K}{\varepsilon} \frac{T_2^2}{\beta_2}$$

whence

$$ln \frac{p_0 K}{\varepsilon} = \frac{T_2\, ln \frac{T_2^2}{\beta_2} - T_1\, ln \frac{T_1^2}{\beta_1}}{T_1 - T_2} \tag{4}$$

Denoting

$$ln \frac{p_0 K}{\varepsilon} = a$$

we have

(5) $$p_0 = \frac{\varepsilon}{K} e^a$$

Again, substituting p_0 from (5) into (3) we find for ε

(6) $$\varepsilon = KT_m \left(ln \frac{T_m^2}{\beta} + a \right)$$

The idea of finding p_0 from the difference of position of the maximum at two different speeds of heating is due to (Parfianowicz 1954). However, the author has given for the computation of the depth of traps only an approximate formula, whereas, by a subsequent application of this method, we find an expression for ε free of any approximations.

The depths of traps computed by the two methods are given in Table 1. The results agree with 12 per cent.

E. Discussion. In the first method, the error of the value of ε is due mainly to the circumstance that the constant p_0 is known only up to the order of magnitude $p_0 = 10^{8 \pm 1} \text{sek}^{-1}$. The error of ε does not exceed 12 per cent. On the other hand, the experimental error in the determination of the temperature of the maximum $\Delta T_m = 2°\text{K}$. is negligeable here and causes $\Delta \varepsilon = 1$ per cent.

Table I

Phosphors	$\beta_1 = 1$ grad/sec	$\beta_2 = 0.5$ grad/sec	Method I $p_0 = 10^8$ sec^{-1}			Metod II	
	T_1	T_2	U	Z	ε	a	ε
CaS—Bi	153°K		23.45	20.43	0.269 eV		
with Mg	258°K		23.98	20.94	0.465 eV		
(violet)	406°K	295°K	24.43	21.37	0.747 eV	12	0.78 eV
CaS—Bi	153°K		23.45	20.43	0.269 eV		
without Mg	385°K		24.38	21.32	0.707 eV		
(violet)	553°K		24.74	21.67	1.103 eV		
CaS—SrS—BaS—Bi	158°K	154°K	23.48	20.47	0.278 eV	12	0.299 eV
(green)	270°K	263°K	24.02	20.98	0.488 eV	12.5	0.55 eV
	369°K	357°K	24.33	21.28	0.677 eV	7	0.60 eV
CaS—SrS—BaS—Bi	155°K		23,46	20.44	0.273 eV		
(blue)	265°K		24.00	20.96	0.478 eV		
	360°K		24.31	21.25	0.659 eV		
BaS—Cu	147°K		23.41	20.39	0.258 eV		
(orange)	296°K		24.11	21.06	0.537 eV		

In the second method the main error for ε. is connected with the error in determining the temperatures of the maximum T_1 and T_2. The quantity a, to be computed from (4), is very sensitive to these errors. For diffuse, high-temperature peaks, this error amounts to $\Delta T = 2°K$ in the conditions of our experiments. As $\varepsilon = KT_m\left(ln\frac{T_m^2}{\beta} + a\right)$ a simple computation yields, in the case $\Delta T_m = 2°K$. for T in the interval $350° - 550°K$, again an error for ε of 20 per cent.

At low temperatures the error for ε. would be greater for the same ΔT_m: however, in this case the peaks are not so broad, and $\Delta T_m = 1°K$. Thus, in both cases, the values ε are subject to an error of about 20 per cent.

Let us discuss now the errors due to the neglect of the retrapping of electrons. Assuming that the retrapping actually takes place (kinetics of II-nd kind — Curie 1955) and that the probabilities of the retrapping, and return of the electron to the centre of luminescence, are equal, one finds a formula

$$J = \frac{p_0 e^{-\frac{\varepsilon}{KT}}}{v\left(\frac{1}{n_0} + \frac{p_0}{v\beta}\int_{T_0}^{T} e^{-\frac{\varepsilon}{KT}} dT\right)^2} \tag{7}$$

(see Romanowski 1946)

where v is the density of traps. The condition for the maximum (by exciting the phosphor to saturation — Parfianowicz 1954) yields

$$\varepsilon' = KT_m\, ln\frac{2\,p_0\,T_m^2\,K}{\varepsilon\,\beta} \tag{8}$$

whereas, by neglecting the retrapping it was

$$\varepsilon = KT_m\, ln\frac{p_0\,T_m^2\,K}{\varepsilon\,\beta}$$

whence

$$\varepsilon' = KT_m\left(ln\frac{p_0\,T_m^2\,K}{\varepsilon\,\beta} + ln\,2\right). \tag{9}$$

Since $ln\frac{p_0\,T_m^2\,K}{\varepsilon\,\beta}$ is of the order 20 and $ln\,2$ of the order 0.7, ε' is larger than ε by 3.5 per cent.

On the other hand, the value ε computed by the second method is independent of whether one starts with formula (2) or with formula (7) (corresponding to the kinetics of the first or second kind). Applying (8) to the computation of $a' = ln\frac{p_0 K}{\varepsilon}$ we have

$$a' = \frac{T_2\, ln\, 2\frac{T_2^2}{\beta_2} - T_1\, ln\, 2\,\frac{T_1^2}{\beta_1}}{T_1 - T_2} \tag{10}$$

Comparing (10) with (4) we find $a' = a - ln\,2$
but

$$\varepsilon' = KT_m\left(ln\,\frac{T_m^{\,2}}{\beta} + ln\,\frac{p_0\,K}{\varepsilon} + ln\,2\right)$$

or

$$\varepsilon' = KT_m\left(ln\,\frac{T_m^{\,2}}{\beta} + a' + ln\,2\right) \tag{11}$$

Comparing (11) with (6) we see that $\varepsilon' = \varepsilon$

Accknowledgement. The author is indebted to Professor A. Jabłoński for suggesting this investigation and for his interest in this paper.

КРАТКОЕ СОДЕРЖАНИЕ

А. Вржесинская, *Активированные кристаллические фосфоры; их производство и кривые термолюминесценции.*

В поисках методов получения ярко светящихся фосфоров из находящихся в продаже материалов произведены следующие наблюдения: а) Чтобы получить ярко светящийся фосфор, следует материал, предназначенный для обжигания, вложить в печь, имеющую уже заранее соответствующую температуру, чтобы получилось стремительное, а не медленное, его нагревание. в) Некоторые плавни, имеющиеся в рецептах в литературе, можно с большим успехом заменить другими. Получены следующие ярко светящиеся фосфоры: 1) CaS-Bi, светящийся фиолетовым светом, 2) фосфорная смесь в составе: CaS, SrS, BaS-Bi, светящаяся зелёным светом или голубым, в зависимости от взаимоотношения выступающих в ней основных веществ, 3) фосфор BaS-Cu, светящийся оранжевым светом, 4) фосфор CaS-Tl, светящийся зелёным, более слабым, чем предыдущий, светом.

Для исследования кривых термолюминесценции сооружена такая аппаратура, благодаря которой кривая накаления чертится сразу посредством светильно-напряжённого пятнышка на фотографической бумаге. Однообразие возростания температуры контролируется специальным устройством. При помощи этой аппаратуры получены кривые накаления фосфоров, изготовленных в первой части этой работы. Произведены некоторые наблюдения относительно роли примеси магния в структуре электронных ловушек.

Глубину ловушек находили с положения (в температурном масштабе) максимум кривых термолюминесценции. Применены при этом два метода. При первом находими глубину ловушек, решая уравнение на максимум кривой термолюминесценции способом графическим. (Нужная для конкретного исчисления глубины ловушки величина постоянной, характерной для определённой группы фосфоров, взята из литераруры. Здесь дело в постоянной p_0, выступающей в формуле для вероятности освобождения электрона из ловушки посредством тепловой энергии: Е. р. $= p_0\,e^{-\frac{\varepsilon}{KT}}$.

Метод второй состоит в применении (внушённого Парфиановичем 1954) метода отыскания постоянной p_0 из двух температур T_1 и T_2, отвечающих тому же самому максимум кривой термолюминесценции при разных скоростях нагревания. Применяя последотельно этот метод, получена точная формула для исчисления глубины ловушки. Результаты, полученные посредством обоих методов сходны в пределах экспериментальных ошибок. Оказывается при этом, что величины для глубины ловушки, полученные методом впервым, не зависят от того, будет ли взята для кривой накаления формула, отвечающая кинетике первого вида (кинетике мономолекулярной), или кинетике второго вида (кинетике бимолекулярной).

REFERENCES

Antonow-Romanowski, *Izw. A. N. SSSR.* 10 (1946).
Curie D., *Journal de Phys.* (1955).
Garlick, *Luminescent Materials.* Oxford (1949).
Lenard-Schmidt Tomaschek, *Hand. d. Exp. phys.*, **23**, 317 (1928).
Parfianowicz, *Zur. exp. teor. fiz.* **26**—6, 697 (1954).
Randall J., and Wilkins M., *Pros. of Roy. Soc.*, **184**—**A**, 347 (1946).
Zirow, *Luminofory*, Moskwa (1950).

ON RELATIVISTIC WAVE EQUATIONS WITH A MASS SPECTRUM[1]

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(Received January 19, 1956)

The question of relativistic wave equations having a mass spectrum and containing new continuous variables is discussed. Mostly the case for which the new variables are components of some space-like four-vector is considered, since similar equations have been very often used in recent times in connection with attempts to build a non-local field theory.

In recent years, in a number of papers on non-local field theory, relativistic wave equations with a mass spectrum and containing new continuous variables have been discussed. Usually, however, no attention is paid to a number of difficulties and characteristic peculiarities connected with the use of these equations. This is why it seems to us worthwhile to publish the present paper devoted to equations of the above-mentioned type.

After the introductory § 1, we will examine in § 2 a number of relativistic wave equations with a mass spectrum which have been used till now. In § 3, some of their general properties will be discussed.

§ 1. Already about 15 years ago, on the basis of some physical considerations which we will not discuss here, (see [1]), investigation was begun of relativistic wave equations with a mass spectrum. The latter means that the equations have solutions corresponding not to one, but to two or more values of rest mass. The equations considered at first were $(\frac{1}{2}-\frac{3}{2})$ and $(1-2)$, corresponding to a particle which can be found in two states with different masses and with spins $\frac{1}{2}$ and $\frac{3}{2}$ respectively, or 1 and 2 respectively. The first of these equations is of interest to us as a generalized model of a nucleon and was therefore more extensively investigated [1, 2, 3]. Subsequently, equations in general form of the type

$$L_k \frac{\partial \Psi}{\partial \chi_k} + i\varkappa \Psi = 0 \tag{1}$$

[1] This paper has been prepared on the basis of a lecture delivered by the author on April 4, 1955, at the All-Union Conference on Quantum Electrodynamics and Theory of Elementary Particles. The appearance of this paper in a Polish Physical journal seemed appropriate, because of the great attention paid by Polish physicists to non-local field theory and to the above-mentioned class of relativistic wave equations.

were studied [4], where L_k is a finite or infinite matrix and $\varkappa$, a number different from zero. Regarded as equations of this type are the widely used equations for particles of spin 0, $\frac{1}{2}$, equations for particles of spin 1, $\frac{3}{2}$ and higher [5, 6], as well as equations for particles with two values of spin. The general investigation of equations of type (1) revealed a number of interesting features, but in the case of a large or infinite number of values of rest mass, the use of such equations (or more accurately, systems of a large or infinite number of equations) for any practical computations is not possible. In additions, no less important is the fact that in (1) the choice of matrix L_4, which determines the values of the rest mass, is left most arbitrary. Therefore, it was natural to seek a constructive path for finding the relativistic wave equations with mass spectra without the use of matrices, which are very cumbersome to work with even for the equation $(\frac{1}{2}-\frac{3}{2})$. Such a non-matrix approach was already outlined at the end of Reference [1] and as a concrete example was developed in [7] in connection with the introduction of new independent variables in the wave function. In other words, the function ψ is considered to be dependent not only on the coordinates and time x_i $(i = 1, 2, 3, 4)$ and spinor indices, but also on some other four-dimensional quantities, for example, some vector u_i. It is immediately obvious that introduction of similar continuous variables is equivalent to the consideration of infinite-dimensional quantities such as ψ in (1) when L_k is an infinite matrix. A function of the type $\Psi(x_i, u_i)$ can, of course, already be considered a scalar, or in the extreme case, can be considered as having a small number of spinor indices (bispinor, vector), since the simultaneous use of both vector u_i and larger (infinite) numbers of indices would be contrary to the very purpose of introducing now continuous variables.

In [7], as we shall see, nearly all the typical peculiarities of the equations of the class under consideration were elucidated, and by virtue of the difficulties encountered, we practically stopped further work in this direction. However, in a whole series of papers [8–14] which have subsequently appeared, discussion of the relativistic wave equations for the function $\Psi(x_i, u_i)$ began a new, but usually without taking into consideration the difficulties we have referred to. On the other hand, one of the possibilities disregarded in [7] is, perhaps, of interest, and does not deserve to be neglected. All this prompted the author to return once again to the discussion of relativistic wave equations with mass spectra.

§ 2. Let us assume that the wave function Ψ is a scalar dependent on the variables x_i and u_i, that the wave equation is not higher than of the second order with respect to x_i, and finally, that the wave equation does not, of course, contain x_i explicitly. Then this equation will have the form

$$\left\{\frac{\partial^2}{\partial x_i\,\partial x_i} - \varkappa^2 + \Phi\left(u_i\,u_i\,u_i\frac{\partial}{\partial u_i}, \frac{\partial^2}{\partial u_i\,\partial u_i}, \frac{\partial^2}{\partial x_i\,\partial u_i}, u_i\frac{\partial}{\partial x_i}\right)\right\}\Psi(x_i, u_i) = 0, \qquad (2)$$

where Φ is some function of its arguments which satisfies the requirements indicated above. (In some cases supplementary conditions of the type (2) can be imposed on the function).

In [7], only equations of the type (2) not containing the quantities $u_i\, u_i$ were considered. In connection with these the operators

$$M_{ik} = u_k \frac{\partial}{\partial u_i} - u_i \frac{\partial}{\partial u_k}, \Phi = \Phi\left(M_{ik}, \frac{\partial}{\partial x_i}\right) \tag{3}$$

$$M_{ik}\, M_{nm} - M_{nm}\, M_{ik} = M_{nk}\, \delta_{im} + M_{km}\, \delta_{in} + M_{in}\, \delta_{km} + M_{mi}\, \delta_{nk} \tag{3a}$$

were used.

We introduce the coordinates

$$u_1 = r \operatorname{ch} \chi \sin \vartheta \cos \varphi,\ u_2 = r \operatorname{ch} \chi \sin \vartheta \sin \varphi,\ u_3 = r \operatorname{ch} \chi \cos \vartheta$$
$$u_4 = i\, r \operatorname{sh} \chi,\ -\infty < \chi < \infty,\ 0 \leqslant \vartheta \leqslant \pi,\ 0 \leqslant \varphi \leqslant 2\pi \tag{4}$$

where the vector u_i is considered space-like (for the possibility of choosing u_i as a time-like vector, see discussion below).

In coordinates (4)

$$\frac{\partial^2}{\partial u_i\, \partial u_i} \equiv \sum_{i=1}^{4} \frac{\partial^2}{\partial u_i^2} = \frac{1}{r^3} \frac{\partial}{\partial r}\left(r^3 \frac{\partial}{\partial r}\right) + \frac{L_1}{r^2}$$

$$L_1 = -\frac{1}{\operatorname{ch}^2 \chi} \frac{\partial}{\partial \chi}\left(\operatorname{ch}^2 \chi \frac{\partial}{\partial \chi}\right) + \frac{\Delta_{\vartheta, \varphi}}{\operatorname{ch}^2 \chi},$$

$$\Delta_{\vartheta, \varphi} = \frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta}\left(\sin \vartheta \frac{\partial}{\partial \vartheta}\right) + \frac{1}{\sin^2 \vartheta} \frac{\partial}{\partial \varphi^2}.$$

$$u_i\, u_i = r^2,\ u_i \frac{\partial}{\partial u_i} = r \frac{\partial}{\partial r},\ \frac{1}{2} M_{ik}\, M_{ik} = u_i\, u_i \frac{\partial^2}{\partial u_k\, \partial u_k} -$$

$$- u_i\, u_k \frac{\partial^2}{\partial u_i\, \partial u_k} - 3 u_i \frac{\partial}{\partial u_i} = L_1 \tag{5}$$

It is obvious from (5) that the operator $\frac{1}{2} M_{ik}\, M_{ik}$ does not contain the variable r, by virtue of which equations containing this operator are known to be compatible with the supplementary condition $(u_i\, u_i - r^2)\, \Psi(x_i\, u_i)$. The same is true of the operators M_{ik} themselves, so that in the sequel in connection with equations containing u_i and $\partial/\partial\, u_i$ only in the form M_{ik}, the above mentioned supplementary condition will not even be explicitly mentioned.

First of all, we shall examine the equation

$$\left(\frac{\partial^2}{\partial x_i\, \partial x_i} - \varkappa^2 + \frac{\beta}{2} M_{ik}\, M_{ik}\right) \Psi(x_i, u_i) = 0 \tag{6}$$

In this case the variables may be separated and obviously

$$\Psi(x_i, u_i) = \psi(x_i)\, \varphi(u_i),\ \left(\frac{\partial^2}{\partial x_i\, \partial x_i} - \varkappa^2 + \beta\, \lambda_1\right) \Psi(x_i) = 0 \tag{7}$$

$$L_1\, \varphi \equiv \frac{1}{2} M_{ik}\, M_{ik}\, \varphi = \lambda_1\, \varphi \tag{8}$$

and the rest energy $E_0 = m_0$ is equal to (everywhere below $h/2\pi = c = 1$)

$$E_0^2 \equiv m_0^2 = \varkappa^2 - \lambda_1 \varphi \tag{9}$$

As solutions of equation (8) we look for functions whose square is integrable on the hypersurface $r^2 = \text{const}$, i. e. we require that the integral $\int |\varphi|^2 \, dS$ be finite, where $dS = \text{ch}^2 \chi \sin \vartheta \, d\chi \, d\vartheta \, d\varphi$ is an element of area on the hypersurface. In [7] such a solution of equation (8) was found directly, but we have no cause to dwell on this, since the eigenvalues of equation (8) can be found in still more general form by using the results of Reference [4]. As already shown in [7], the eigenvalues λ_1 and λ_2 of the operators L_1 and $L_2 = -\frac{1}{2} i \varepsilon_{iklm} M_{ik} M_{lm}$, where ε_{iklm} is the completely anti-symmetric unit pseudo-tensor, are equal to

$$\lambda_1 = -(j^2 - 1) + \alpha^2,$$
$$\lambda_2 = -\alpha j, \tag{10}$$

where j is an integer or half-integer and α is a real, positive number, and where for every j there are an infinite number of solutions differing by whole numbers $l \geqslant j$, $|m| \leqslant l$ (here $l(l+1)$ and m are the eigenvalues of the operators $-(M_{23}^2 + M_{31}^2 + M_{12}^2)$ and $i M_{12}$). It is assumed that the operators M_{ik} are in reality operators satisfying relations (3a) and that they can be obtained in various ways. In the particular case where they have been obtained[1] in the form (3), it is easy to see that $L_2 = 0$ and therefore for $j \neq 0$, $\alpha = 0$ and that $\alpha \neq 0$ only when $j = 0$. Further, in the case of the space-like u_i, the value $j = 0$ is not permissible (the square of function φ is not integrable) and thus the solution of equation (8) is

$$\lambda_1 = -(j^2 - 1), \quad j = 1, 2, 3, \ldots, \; l = j, j+1, \ldots \tag{11}$$

On the other hand, if vector u_i is time-like, equation (8) will have only the solution $j = l = 0$, which corresponds to a continuous spectrum $\lambda_1 = \alpha^2$. Below, we will always assume that vector u_i is space-like.

Returning to equation (6), we see that the mass spectrum (9) has the form $E_0^2 \equiv m_0^2 = \varkappa^2 + \beta(j^2 - 1)$ and is infinitely degenerate (since m_0 does not depend on the number $l \geqslant j$). Solutions of equation (8) are connected with infinite dimensional unitary representations of the Lorentz group which are considered in detail in Ref. [4]. Since the number l is associated with the spin of the particle, the degeneracy of the l-spectrum means that the spin of the particle in a state with a given mass is indeterminate and that equation (6) obviously cannot provide a true picture. There

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[1] As example of a realization different from (3) the following one may be mentioned

$$M_{ik} = S_{kl} \frac{\partial}{\partial S_{il}} - S_{il} \frac{\partial}{\partial S_{kl}},$$

where S_{ik} is an antisymmetric tensor.

are two ways in which the degeneracy can be removed. The first consists of imposing upon Ψ some supplementary condition such as the equation used in Ref. [7]:

$$\left\{M_{il}\, M_{kl}\, \frac{\partial^2}{\partial x_i \partial x_k} - (j_0 + 1)\, \frac{\partial}{\partial x_i\, \partial x_k}\right\} \Psi(x_i, u_i) = 0 \tag{12}$$

In the rest system, this equation has the form $[M_{4l}\, M_{4l} - (j_0 + 1)]\, \Psi = 0$ and its solution satisfying (6) is different from zero if

$$l^2 + l + j^2 - j_0 = 0\,,\ l \geqslant j, j \neq 0;\ l \text{ and } j \text{ — integers.} \tag{13}$$

Equation (13), in turn, has a solution only when j_0 which was introduced in (12), is an integer; in this case, for a given j_0 there exists one or more solutions corresponding to different spins and rest masses (for details, see Ref. [7]. The rather chance character of the spectrum obtained and, above all, the difficulties of introducing the interaction — of which we will speak in § 3 — make it of greater interest to investigate the second path. This path consists of introducing into the equation mixed terms containing M_{ik} and $\frac{\partial}{\partial x_i}$. Thus in [7], there was a discussion of the equation

$$\left\{\frac{\partial^2}{\partial x_i\, \partial x_i} - \varkappa^2 + \frac{\beta}{2}\, M_{ik}\, M_{ik} + \varepsilon\, M_{il}\, M_{kl}\, \frac{\partial^2}{\partial x_i\, \partial x_k}\right\} \Psi(x_i, u_i) = 0 \tag{14}$$

which leads to the spectrum

$$E_0^{\,2} \equiv m_0^{\,2} = \frac{\varkappa^2 - \beta\,(a^2 - j^2 + 1)}{1 + \varepsilon\,(l^2 + l - j^2 + a^2 + 1)}\,, \tag{15}$$

where it should be assumed that for the space-like vector u_i, $a = 0$ (this case will always be implied below). For $\beta > 0$ and $\varepsilon > 0$ the spectrum (15) has a point of condensation for $m_0 = 0$. If, for example, $\beta > 0$, but $\varepsilon < 0$, then there will be solutions with $m_0^{\,2} < 0$, corresponding to functions of the type $\Psi = e^{\pm |m_0| t}$. Such solutions should, of course, be discarded. In [7] however, a more fundamental conclusion of the necessity of neglecting the equations leading to imaginary masses was given (we shall return to this in § 3). Thus we abandoned further investigation of equations of type (14), since it turned out that they always had either a decreasing spectrum or allowed solutions with imaginary masses[2].

The relativistic equations for scalar or bispinor functions $\Psi(x_i, u_i)$ which were subsequently examined in a number of other papers [8—14] also possess the charac-

[2] Included in this group are the equations considered in [7], namely

$$\left(\gamma_i \frac{\partial}{\partial x_i} + b\gamma_i \gamma_k\, M_{ik} + \varkappa\right) \Psi(x_i, u_i) = 0\,,$$

$$\frac{\partial^2}{\partial x_k\, \partial x_k}\, \Psi_i - \varkappa^2\, \Psi_i + b\, M_{ik}\, \Psi_k + \varepsilon\, \frac{\partial}{\partial x_i}\, \frac{\partial\, \Psi_k}{\partial x_k} = 0$$

where, in the first case, Ψ is a bispinor, and, in the second, a 4-vector.

teristic peculiarities noted above or have solutions corresponding to zero rest mass. As we have already pointed out in [15], the well known equations introduced by Yukawa [8–9] are entirely of type (2). Indeed, the following equations were considered in [8]

$$\left\{\frac{\partial^2}{\partial x_i\,\partial x_i} - \varkappa^2\right\}\Psi(x_i, u_i) = 0, \quad \{u_i\,u_i - r^2\}\,\Psi = 0, \quad u_i\frac{\partial}{\partial x_i}\Psi = 0 \tag{16}$$

It is quite obvious, from what was said above, that the system (16) corresponds to a particle with an indeterminate spin and not to a group of particles with spin zero, as, at first, assumed in [8]. In [9], the following equition was discussed

$$\left\{\frac{\partial^2}{\partial x_i\,\partial x_i} - \frac{\mu^2}{2}\left[-\frac{\partial^2}{\partial u_i\,\partial u_i} + \frac{1}{\mu^4}\,u_i\,u_i\right]^2\right\}\Psi(x_i, u_i) = 0 \tag{17}$$

where in [9], μ was denoted by λ and u_i by r_μ.

Making the substitution $\Psi(x_i, u_i) = \psi(x_i)\,\varphi(u_i)$ we are lead to the solution

$$\varphi(u_i) = H_{n_1}(u_1/\mu)\cdot H_{n_2}(u_2/\mu)\cdot H_{n_3}(u_{n_3}/\mu)\cdot H_{n_0}(u_0/\mu)\times$$
$$\times\exp\left\{-\frac{u_1^2 + u^2{}_2 + u^2{}_3 + u^2{}_0}{2\mu^2}\right\} \tag{18}$$

$$m_0{}^2 = \frac{2}{\mu^2}(n_1 + n_2 + n_3 - n_0 + 1)^2, \qquad n_{0,1,2,3} = 0,1,2,\ldots.$$

where $u_0 = i\,u_4$ and H_{ni}, is a Hermite polynomial.

Spectrum (18) is infinitely degenerate and contains values of zero for the rest mass. In order to remove the degeneracy, in [9], an equation containing mixed terms[3] was considered, namely

$$\left\{\frac{\partial^2}{\partial x_i\,\partial x_i} - \varkappa^2 - \frac{\mu^2}{2}\left[-\frac{\partial^2}{\partial u_i\,\partial u_i} + \frac{1}{\mu^4}\,u_i\,u_i\right]^2 + \right.$$
$$\left. + \varepsilon\,\mu^2\left[-\left(\frac{\partial^2}{\partial x_i\,\partial u_i}\right)^2 + \frac{1}{\mu^4}\left(u_i\frac{\partial}{\partial x_i}\right)\right]\right\}\Psi = 0 \tag{19}$$

Solving this equation in the rest system, we obtain the mass spectrum

$$m_0{}^2 = \frac{\varkappa^2 + \dfrac{2}{\mu^2}(n_1 + n_2 + n_3 - n_0 + 1)^2}{1 - 2\,\varepsilon\left(n_0 + \dfrac{1}{2}\right)} \tag{20}$$

According to the sign of ε, as in the case of spectrum (15), this spectrum either contains a decreasing branch ($m_0 \to 0$) or leads to the appearance of imaginary masses.

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[3] In equation (19) of [9], $\varepsilon = \beta^2$ and $\varkappa^2 = 0$. We introduce the value $\varkappa^2 \neq 0$ in order to get rid of a solution of the type $\Psi(x_i, u_i) = \varphi(u_i)$.

In [10], the following system of equations was used

$$\left(\gamma_i \frac{\partial}{\partial x_i} - 2\gamma_i' \frac{\partial}{\partial u_i} + \varkappa\right) \Psi(x_i, u_i) = 0$$

$$u_i \frac{\partial}{\partial x_i} \Psi = 0, \quad \left(\lambda^2 \frac{\partial^2}{\partial x_i \partial x_i} - \lambda^2 u_i u_i\right) \Psi = 0 \tag{21}$$

which, however, is not a consistent system.[4]

There is no need to dwell in detail on this point or on other critical observations made by us earlier [16] in relation to Ref. [10] sin e their basic features were subsequently discussed by Rayski himself in [11].

In [11], the following system of equations is examined

$$\left\{\frac{\partial^2}{\partial x_i \partial x_i} - \varkappa^2 - 4C\left[-\frac{\partial^2}{\partial u_i \partial u_i} + \frac{1}{u_i u_i}\left\{\lambda^4 \left(\frac{\partial^2}{\partial x_i \partial u_i}\right)^2 + u_i u_k \frac{\partial^2}{\partial u_i \partial u_k} + \right.\right.\right.$$

$$\left.\left.\left. + 2u_i \frac{\partial}{\partial u_i}\right\}\right]\right\} \Psi(x_i, u_i) = 0, \; u_i \frac{\partial \Psi}{\partial u_i} = 0, \; \left(\lambda^2 \frac{\partial^2}{\partial x_i \partial x_i} - \lambda^{-2} u_i u_i\right) \Psi = 0 \tag{22}$$

This system is a consistent one and leads to the spectrum

$$m_0^2 = \frac{\varkappa^2}{2} \pm \sqrt{\frac{\varkappa^4}{4} + 4C \frac{l(l+1)}{\lambda^4}}, \qquad l = 0, 1. 2, \ldots \tag{23}$$

Here we again meet with imaginary masses as well as with differential supplementary conditions.

The following equation was suggested in [13]

$$\left\{\frac{\partial^2}{\partial x_i \partial x_i} - \frac{M_{jl} M_{kl} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k}}{\lambda^2 \frac{\partial^2}{\partial x_i \partial x_i}}\right\} \Psi(x_i, u_i) = 0 \tag{24}$$

where M_{ik} is the operator (3).

An analogous equation containing the operator $\partial^2/\partial x_i \partial x_i$ was also suggested in [12, 14]. In such cases, we are no longer dealing with second order differential equations with respect to x_i, to which we are restricting ourselves. We shall note only that if both parts of equation (24) are multiplied by $\partial^2/\partial x_i \partial x_i$, then we obtain a 4th order equation, two solutions of which correspond to zero rest mass. That is why

[4] In [10] as well as in the analogous case in [12], an attempt was made to examine the supplementary conditions of the type $u_i (\partial/\partial x_i) \Psi = 0$ not as a new equation for Ψ, but as a condition for limiting the region of variability of the variable u_i (for example, from $u_4 = 0$, it was concluded that $(\partial/\partial u_4) \Psi = 0$. It is not obvious to us if such an approach is, in general, permissible in relativistic theory, and in any case, there is no doubt that such a procedure extends outside the framework of the general scheme [4] and meets with difficulties when an interaction with the field is introduced. We shall not discuss such attempts in detail, and we note only that with such an approach, it is obviously not allowed in any case to proceed from the system (21), which has a quite definite mathematical meaning and is not a consistent system.

even a transition to a 4th order equation, which is still not a particularly drastic change (it is simply associated with the doubling of the number of states), is not of interest to us in the given case if we are not dealing with particles having a zero rest mass. As regards equations of type (24), which contain derivatives in the denominator, it is not clear to us, in general, how to work with them.

Finally, we will consider the system of equation suggested in [14] (in (25) Ψ is a bispinor)

$$\left\{\gamma_i \frac{\partial}{\partial x_i} + \varkappa + a\left(-\frac{\partial^2}{\partial u_i\, \partial u_i} + u_i\, u_i\right)\right\} \Psi(x_i, u_i) = 0$$

$$\frac{\partial}{\partial x_i}\left(u_i - \frac{\partial}{\partial u_i}\right)\Psi = 0 \tag{25}$$

This system is a consistent one and leads to the spectrum

$$m_0 = \pm\,[\varkappa + a\,(n_1 + n_2 + n_3 + 1)], \tag{26}$$

which does not possess any of the „defects" of the imaginary or zero mass type.

§ 3. All the equations introduced possess characteristic peculiarities which we already mentioned in discussing Ref. [7]. These particuliarities are as follows:

If we have one equation of type (2) in which there are no mixed terms (as regards variables x_i and u_i), then the spectrum will be infinitely degenerate (see (6)—(11), (17)). The case of one equation of type (2) is then related to the general scheme [4] based on equations of class (1).

If we have several equations of type (2) relating to one function Ψ being differential equations[5] with respect to x_i, then the system is overdefined and extends outside the framework of the scheme [4]. At the same time, such a system can have non trivial solutions in which the degeneracy, generally speaking, disappears. The systems of equations (6) and (12), (22) and (25) are examples of this; however, in the case of system (16), the degeneracy is preserved. As in § 2, we will call systems of two or more equations associated with one function Ψ, equations will supplementary conditions, even though the basic equation and the supplementary condition sometimes differ from each other very little. Equations with supplementary conditions can possess a quite "good" spectrum, i. e., for example, a spectrum increasing to infinity with finite degeneracy (see (25)—(26)) and having neither an imaginary mass nor a rest mass close to zero. However, equations with supplementary conditions possess a considerable shortcoming — the field cannot be introduced in them by the usual simple method. The situation here is similar to the case of particles with spins $\frac{3}{2}$ or higher [5, 6] for which it is not permissible to introduce the electromagnetic field into the equation by changing $\partial/\partial x_k \to \partial/\partial x_k - i\,e\,A_k$ as it makes the system inconsistent. For

[5] In referring, hereafter, to supplementary conditions, we shall have in mind differential supplementary conditions, and not conditions of the type $(u_i u_i - r^2)\,\Psi = 0$.

particles with spins $\frac{3}{2}$, 2 ($\frac{1}{2} - \frac{3}{2}$) and so on, this difficulty can be circumvented by introducing new functions and a variational formulation of the problem [5, 6, 1]. It is highly probable that an analogous approach is also possible for equations of type (2) with supplementary conditions of the same type. However, this problem has not yet been solved. At the same time, even if a solution to this problem is found, equations with supplementary conditions will be very complicated if interactions are present. Therefore, the possibility of working without the supplementary conditions is especially attractive.

The degeneracy can be removed not only by imposing supplementary conditions, but also by making a transition to equations of type (2) with mixed terms. In this case, if imaginary masses do not appear, then in all the known examples (see (14) – (15) and (19) – (20)), there appears a branch of the mass spectrum which tends to zero. This result has a very general character, as it is true [4], generally speaking, for the case of any equation of type (1), which also includes equations of type (2) with mixed terms but without supplementary conditions. It is thus clear that a non-degenerate increasing mass spectrum with one equation of type (2) can be obtained only in the presence of solutions with imaginary masses[6]. Solutions with imaginary masses $\Psi \sim \exp\{\pm |m_0| t\}$ should, of course, be discarded, but it is not yet clear if this means that the corresponding equation itself is unsuitable. In [7], such equations were nevertheless rejected, but in [9] only the inadmissible solutions were discarded, while the equations were retained.

Up to the present time, in all applications to elementary particles, the imaginary masses do not appear. On the other hand, the rejection of one or another solution is not an unusual procedure in quantum theory, and, in particular, the discarding of solutions with imaginary masses can be treated simply as a generalization of the requirement for the finiteness of the solutions. Indeed, the solution with $E^2_0 = m^2_0 < 0$ and momentum $p = 0$ under a transition to a moving system of reference corresponds to a solution which has an imaginary momentum and which does not remain finite in space. Therefore, by introducing an additional requirement, we come to discard only solutions with $p = 0$, which increase without limit as $t \to \pm\infty$ and which are obtained by a Lorentz transformation from inadmissible solutions with imaginary momenta. By virtue of the above, the discarding of equations with imaginary masses in [7,4] does not seem well-founded. However, one cannot also have full confidence in using such equations, since the presence of the discarded imaginary solutions can "take revenge" when we deal with the interaction of the particle with the field. In addition to this feature, to which we shall still return, it should be noted that the presence

[4] We note that the statement about the appearance of a decreasing spectrum for equations of type (1) represents the rule, but by way of exception an increasing spectrum can also be obtained (see, for example, (9) and (11)). This spectrum, however, is infinitely degenerate. It is probably not possible to obtain a non-degenerate increasing spectrum (without the presence of imaginary masses), although this statement has not been strictly proved.

of imaginary masses usually accompanies also the appearance of space-like solutions (solutions with space-like momenta p_i [16]).

Let us examine, for example, the equation

$$\left(\frac{\partial^2}{\partial x_i\,\partial x_i} + \varkappa^2\right)\psi(x_i) = 0, \tag{27}$$

which in the rest system has solutions with imaginary masses, since $E^2{}_0 \equiv m_0{}^2 = -\varkappa^2$. We discard these solutions and other solutions with imaginary E and p as a result of applying the generalized principle of finiteness of solutions (see above), but this equation will still have solutions with real E and p satisfying the relation

$$E^2 = -\varkappa^2 + p^2. \tag{28}$$

Such solutions for which $p^2 \geqslant \varkappa^2$ are everywhere finite and it is impossible to discard them on the grounds of the required finiteness. On the other hand, these space-like solutions correspond to velocities greater than the velocity of light ($v = \partial E/\partial p = p/\sqrt{p^2 - \varkappa^2} \geqslant 1$) and are obviously inadmissible because of physical considerations. The simultaneous appearance of imaginary masses and space-like solutions is the rule, and it occurs, perhaps, for all equations having a finite number of dimensions (this problem has not been investigated in detail). However, for equations with an infinite number of dimensions, cases exist where there are solutions corresponding to imaginary masses, but no space-like solutions. For example, equation (19) for $E = 0$, $p_2 = p_3 = 0$ and $p_1 \neq 0$, has the solution

$$p_1{}^2 = \frac{\varkappa^2 + \dfrac{2}{\mu^2}(n_1 + n_2 + n_3 - n_0 + 1)^2}{1 + 2\varepsilon\left(n_0 + \dfrac{1}{2}\right)} \tag{29}$$

If $\varepsilon > 0$, equation (19) has solutions with imaginary masses (see (20)), but in this case $p^2{}_1 < 0$ everywhere and therefore there are no space-like solutions for which p^2 would be greater than zero and E equal to zero. On the orther hand, for many other equations, space-like solutions appear. This occurs, for example, if in (19) the third term is equal to

$$-\frac{\mu^2}{2}\left(\frac{\partial^2}{\partial u_i\,\partial u_i} + \frac{1}{\mu^4}\,u_i\,u_i\right)$$

and so on.

As shown in [17] space-like solutions also appear for the system of equations (22) considered by Rayski [11]; in this case there are also solutions with zero mass. All similar equations containing space-like solutions apparently should be considered inadmissible.

As regards equations with imaginary masses, but without space-like solutions, there is, as already pointed out, no basis for discarding them as long as we are dealing with free equations. Thus some equations with imaginary masses and also individual equations with supplementary conditions obviously lead to increasing mass spectra with non-infinite degeneracy. In addition, one can apparently propose as many such

equations as one wants by modifying the corresponding form of function Φ in (2). For these reasons, we somewhat sceptically regard the attempts made [11] to connect one or another equation of type (2) with the mass spectrum of known particles; if we are not to rely simply on a lucky guess for choosing some equation out of all possible equations, then we need a new idea which so far has not been enunciated[8].

Despite the fact that it is difficult nowadays to connect the relativistic wave equations with mass spectra with well-founded hopes for progress in the sphere of the physics of elementary particles, further investigation of these equations would be of interest, if even from methodical considerations.

Thus the centre of attention undoubtedly should be shifted to the examination of equations with interaction and investigation of the solutions of those equations, even though in the simpler cases (constant external magnetic field, onedimensional electric field, first-order radiation processes, and so on). In speaking of interaction, we have in mind here the usual local interaction, the study of which, one would think, should precede the introduction of the non-local interaction, since the examination of the latter is accompanied by additional difficulties.

In our opinion the above remarks apply in even greater measure to the case in which one considers infinitely degenerate equations in the hope that the interaction with the field will remove the degeneracy [9]. In such conditions, perturbation theory is not applicable (as it cannot, generally speaking, lead to the appearance of new values of mass) and the interaction problem should be solved by more accurate methods. But the application of such methods, as it is known, meets with great difficulty even in the case of local interaction with the simplest fields and the transfer of these methods to the equations interesting us is by itself an independent, very difficult problem even for local interaction. Finally, we note that the presence of strong interactions makes it possible to use the free equations for finding the mass spectrum only under the assumption that the change of mass due to the interaction vanishes or, at any rate, is very small. Such an assumption, however, is very unlikely if we are dealing with a theory without divergencies where there are no grounds for renormalization, or simply leaving out of account all the field contributions to the self-energy of the particle. Thus,

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[7] If new tensor arguments of the type S_{ik} are introduced into function Ψ, then, generally speaking, a continuous mass spectrum is obtained. By considering Ψ to be a function of some scalar, depending on the form of function Φ in the equation $\left\{\frac{\partial^2}{\partial x_i\,\partial x_i} - \varkappa^2 + \Phi\left(u, \frac{\partial}{\partial u}\right)\right\}\Psi(x_i, u) = 0$, any spectrum can be obtained. The introduction of a scalar argument is particularly simple and does not cause any difficulties, but on the other hand, it can hardly be of any interest in connection with physical applications.

[8] We are now making no mention of the fact that in recent times there has been no convincing basis for taking the spin of mesons and hyperons to be larger than 0 and $1/2$. If the spins of all particles are equal to zero or $1/2$, the mass spectra probably cannot be obtained by considering variables of the type u_i, which although not spin variables in the literal sense of the word are clearly connected with the spin of the particle.

the possibility of comparing the spectra of the free equations with the mass spectra of real particles becomes still more illusory.

We shall leave, however, the discussion at this point, since the principal purpose of the present paper is only to cast some light on the present state of the problem concerning the properties and character of free relativistic wave equations with mass spectra.

КРАТКОЕ СОДЕРЖАНИЕ

В. Л. Гинзбург, *О релятивистских волновых уравнениях со спектром масс.*

В последние годы в ряде работ обсуждаются релятивистские волновые уравнения со спектром масс, содержащие новые непрерывные независимые переменные. При этом, однако, не обращается обычно внимания на ряд затруднений и особенностей, связанных с исползованием таких уравнений.

Релятивистские волновые уравнения, зависящие от новых переменных, в большинстве случаев от некоторого пространственно-подобного четырехмерного вектора u_i, подробно рассматривались в работе [7], а затем и в ряде других статей [8—17]. Таковы, в частности, уравнения (2), (6), (14), (16), (17), (19), (22), (24) и (25), приведенные в тексте. Все эти уравнения имеют следующие особенности:

Если имеется одно уравнение типа (2), причем в нем нет членов смешанных относительно переменных x_i и u_i, то спектр масс бесконечно вырожден (см. (6) — (11) и (17)). Случай одного уравнения типа (2) относится при этом к общей схеме [4] базирующейся на уравнениях класса (1).

Если имеется несколько дифференциальных уравнений типа (2) относительно одной функции $\Psi(x_i, u_i)$, то система выходит за рамки схемы [4] и, вообще говоря, переопределена. Однако, подобные системы могут иметь нетривиальные решения, причем вырождение обычно снимается (см. системы (6) и (12), (22) и (25)). Основной недостаток этих уравнений, которые мы называем уравнениями с дополнительными условиями, связан с трудностью ввести в них взаимодействие с другими полями.

Если имеется одно уравнение типа (2), содержащее смешанные по x_i и u_i члены, вырождение спектра также снимается. При этом, если не появляются мнимые массы, то во всех известных случаях имеется стремящаяся к нулю ветвь значений массы покоя (см. (14 — (15) и (19) — (20). Поэтому получение невырожденного растущего спектра масс, на базе одного уравнения типа (2), связано с наличием решений с мнимыми массами. Эти решения должны, конечно, быть отброшены, что еще быть может не делает непригодным само соответствующее уравнение. Однако, в целом ряде случаев появление мнимых решений оказывается связанным с появлением также и пространственно-подобных решений (решений с пространственно-подобным импульсом p_i). Эти решения физически недопустимы, а отбросить их с помощью принципа конечности решений нельзя. Поэтому соответствующие уравнения, к числу которых принадлежат уравнения (22) из [11], повидимому, должны заведомо считаться недопустимыми.

Таким образом только некоторые уравнения с дополнительными условиями и уравнения с мнимыми массами, но без пространственноподобных решений, не вызывают возражений общего характера, пока речь идет только о свободных уравнениях. Не исключено, однако, что при наличии взаимодействия попытка использования и таких „хороших“ уравнений столкнется с новыми серьезными затруднениями. Исследование свойств и решений релятивистских волновых уравнений со спектром масс при наличии взаимодействия с электромагнитным и другими полями представляется нам имеющим несомненный методический интерес. Что же касается использования подобных уравнений в области физики элементарных частиц, то в настоящее время трудно надеяться на успех в этом отношении.

REFERENCES

1. Ginzburg, V. L., *Zh. eksp. teor. Fiz.*, **13**, 33 (1943); *J. Phys. USSR*, **8**, 33 (1944); *Phys. Rev.*, **63**, 1 (1943).
2. Bhabha, H., *Phil. Mag.*, **43**, 33 (1952).
3. Fainberg, V. Y., *Zh. eksp. teor. Fiz.*, **25**, 636, 644 (1953).
4. Gelfand, I. M. and Yaglom A. M., *Zh. eksp. teor. Fiz.*, **18**, 703 (1948).
5. Fierz, M. and Pauli, W., *Proc. Roy. Soc.*, **173**, 211 (1939).
6. Ginzburg, V. L., *Zh. eksp. teor. Fiz.*, **12**, 425 (1942); *J. Phys. USSR*, **7**, 115 (1943).
7. Ginzburg, V. L. and Tamm I. E., *Zh. eksp. teor. Fiz.*, **17**, 227 (1947).
8. Yukawa, H., *Phys. Rev.*, **77**, 219 (1950).
9. Yukawa, H., *Phys. Rev.*, **91**, 415, 416 (1953).
10. Rayski, J., *Nuovo Cimento*, **10**, 1729 (1953).
11. Rayski, J., *Fortschr. Phys.*, **2**, 165 (1954); *Acta phys. Pol.*, **14**, 107 (1955); *Nouvo Cimento* **2**, 255 (1955).
12. Shirokov, Y. M., *Zh. eksp. teor. Fiz.*, **21**, 748 (1951).
13. Hara, O., Marumori, T., Ohnuki, Y., and Shindo, M.. *Prog. Teor. Phys.* **12**, 167 (1954).
14. Markov, M. A., *Dokl. Akad. Nauk* 101, **51** (1955).
15. Ginzburg, V. L., *Zh. eksp. teor. Fiz.*, **25**, 757 (1953).
16. Ginzburg, V. L. and Silin, V. P., *Zh. eksp. teor. Fiz.* **27**, 116 (1954).
17. Zhirnov, V. A. *Zh. eksp. teor. Fiz.* **30**, 425 (1956).

SOME REMARKS ON THE THEORY OF THE ELECTRON — PHOTON CASCADE

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(Received February 16, 1956)

The well known integro-differential equations for the first moments of the distribution for the electron — photon cascade are derived from a integral equation analogous to the Smoluchowski equation from the theory of stochastic processes. Some asymptotic properties of the distribution function are given.

1. Introduction

This note contains some preliminary considerations, which are useful for the two papers which follow (Stachowiak 1956, Konwent and Łopuszański 1956). We give the derivation of well known integro-differential equations for the first moments of the electron — photon cascade in approximation *A*. In this derivation, we start from an integral equation, analoguous to the Smoluchowski equation[1] in the theory of stochastic processes. In addition we present some remarks on the behaviour of the distribution function.

2. Notation

We denote by

$$\frac{1}{E_0} s^{(i,k)}(\varepsilon, x) \tag{1}$$

the average density of particles of type k in the energy range $(\varepsilon, \varepsilon + d\varepsilon)$ E_0 at the depth x due to a primary particle of type i and energy E_0. The supperscript $i, k = 1,2$, where we designate the electron by 1 and photon — by 2.

We have further

$$s^{(i,k)}(\varepsilon, x) = -\frac{\partial S^{(i,k)}}{\partial \varepsilon}(\varepsilon, x) \tag{2}$$

where $S^{(i,k)}(\varepsilon, x)$ denotes the average number of particles of type k of energy larger than εE_0 at the depth x due to primary particle of type i of energy E_0.

[1] This equation is also called the Chapman — Kołmogorow equation.

3. Derivation of the equations

The starting point of our considerations is the integral equation

$$s^{(i,k)}(\varepsilon, x) = \int_0^1 \sum_{l=1}^{2} s^{(i,l)}(\varepsilon', x')\, s^{(l,k)}\left(\frac{\varepsilon}{\varepsilon'}, x - x'\right)\frac{d\varepsilon'}{\varepsilon'} \tag{3}$$

for $x \geqslant x' \geqslant 0$.

We write the following obvious relations

$$s^{(l,k)}(\varepsilon, 0) = \delta_{lk}\,\delta(1-\varepsilon) \tag{4a}$$

$$\left(\frac{\partial s^{(l,k)}(\varepsilon, x)}{\partial x}\right)_{x=0} = \begin{cases} -\delta(1-\varepsilon)\,\alpha^{(1)} + w^{(1)}(1-\varepsilon) & \text{for } l = k = 1 \\ w^{(1)}(\varepsilon) & \text{for } l = 1,\ k = 2 \\ 2w^{(2)}(\varepsilon) & \text{for } l = 2,\ k = 1 \\ -\delta(1-\varepsilon)\,\alpha^{(2)} & \text{for } l = k = 2 \end{cases} \tag{4b}$$

where $w^{(1)}(\varepsilon)$ and $w^{(2)}(\varepsilon)$ denote the bremsstrahlung and pair production differential cross sections respectively and

$$\alpha^{(1)} = \int_0^1 w^{(1)}(\varepsilon)\, d\varepsilon, \qquad \alpha^{(2)} = \int_0^1 w^{(2)}(\varepsilon)\, d\varepsilon.$$

We differentiate both sides of equation (3) with respect to x and put $x' = x$. Taking into account relations (4), we obtain

$$\frac{\partial s^{(i,1)}(\varepsilon, x)}{\partial x} = -\alpha^{(1)} s^{(i,1)}(\varepsilon, x) + \int_0^1 s^{(i,1)}(\varepsilon', x) \times$$

$$\times\, w^{(1)}\left(1 - \frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} + 2\int_0^1 s^{(i,2)}(\varepsilon', x)\, w^{(2)}\left(\frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} \tag{5a}$$

$$\frac{\partial s^{(i,2)}(\varepsilon, x)}{\partial x} = -\alpha^{(2)} s^{(i,2)}(\varepsilon, x) + \int_0^1 s^{(i,1)}(\varepsilon', x)\, w^{(1)}\left(\frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} \tag{5b}$$

This is already the first group of well known equtions, which are discussed for the case of boundary conditions (4a) e.g. by Landau and Rumer (1938), Bhabha (1950), Bhabha and Chakrabarty (1943, 1948).

To obtain the second group of equations, we differentiate (3) with respect to x'. Further, we make use of equation (3), differentiated with respect to x, and finally put $x' = 0$.

With help of (4b), we have

$$\frac{\partial s^{(1,k)}(\varepsilon, x)}{\partial x} = -\alpha^{(1)} s^{(1,k)}(\varepsilon, x) + \int_0^1 w^{(1)}(\varepsilon') \left\{ s^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) + s^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) \right\} \frac{d\varepsilon'}{\varepsilon'} \tag{6a}$$

and

$$\frac{\partial s^{(2,k)}(\varepsilon, x)}{\partial x} = -\alpha^{(2)} s^{(2,k)}(\varepsilon, x) + 2\int_0^1 w^{(2)}(\varepsilon') s^{(1,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) \frac{d\varepsilon'}{\varepsilon'} \tag{6b}$$

These equations form the second group of integrodifferential equations.

Integrating (6) into ε in the interval ($\eta < 1{,}1$) we arrive at the well known equations

$$\frac{\partial S^{(1,k)}(\eta, x)}{\partial x} + \alpha^{(1)} S^{(1,k)}(\eta, x) = \int_0^1 w^{(1)}(\varepsilon') \left\{ S^{(1,k)}\left(\frac{\eta}{1-\varepsilon'}, x\right) + S^{(2,k)}\left(\frac{\eta}{\varepsilon'}, x\right) \right\} d\varepsilon' \tag{7a}$$

and

$$\frac{\partial S^{(2,k)}(\eta, x)}{\partial x} + \alpha^{(2)} S^{(2,k)}(\eta, x) = 2\int_0^1 w^{(2)}(\varepsilon') S^{(1,k)}\left(\frac{\eta}{\varepsilon'}, x\right) d\varepsilon' \tag{7b}$$

considered e.g. by Janossy and Messel (1950).

In the first paper following this note extensive use is made of (7). The second paper makes extensive use of (5).

4. Behaviour of the distribution function

We shall now show that $S^{(i,k)}(\varepsilon, x)$ is of the same order of magnitude as the probability of finding at least one particle of type k at depth x and having an energy larger than εE_0 along with an arbitrary number of particles of type $(3-k)$ due to primary particle of type i and energy E_0. We denote this probability by $P^{(i,k)}(\varepsilon, x)$. To show this, we note that

$$\left[\frac{1}{\varepsilon}\right]^{-1} S^{(i,k)} \leqslant P^{(i,k)}(\varepsilon, x) \leqslant S^{(i,k)}(\varepsilon, x) \tag{8}$$

Urbanik (1955) has shown that

$$S^{(i,k)}(\varepsilon, x) = e^{-\alpha^{(2)} x} W^{(i,k)}(\varepsilon, x) \tag{9a}$$

where for each $\beta > 0$

$$\lim_{x \to \infty} e^{-\beta x} W^{(i,k)}(\varepsilon, x) = 0 \tag{9b}$$

and

$$\lim_{x \to \infty} W^{(i,k)}(\varepsilon, x) = \infty \tag{9c}$$

Let us write $P^{(i,k)}$ in the form

$$P^{(i,k)} = e^{-\alpha^{(2)}x} V^{(i,k)}(\varepsilon, x) \tag{10a}$$

Then, from (8) it follows that

$$\lim_{x \to \infty} e^{\beta x} V^{(i,k)} \leqslant \lim_{x \to \infty} e^{-\beta x} W^{(i,k)} = 0 \tag{10b}$$

and further that

$$\lim_{x \to \infty} V^{(i,k)} > \left[\frac{1}{\varepsilon}\right]^{-1} \lim_{x \to \infty} W^{(i,k)} = \infty \tag{10c}$$

Equations (10) represent the generalization of Urbanik's result for the distribution function.

5. *Final remarks*

From (6) and (7) it is obvious that for large depth of the absorber all asymptotic properties deduced from (7) for $S^{(i,k)}$ are also valid, mutatis mutandis, for $s^{(i,k)}$. Further, from (8), it follows that the same is also true for $P^{(i,k)}$.

Consequently the results obtained in the first of the two following papers (which deal with equations (7)) are also valid mutatis mutandis for $s^{(i,k)}$ as well as for $P^{(i,k)}$. In the case of the second paper, which deals with equations (5), a similar situation occurs, but further proof is needed.

КРАТКОЕ СОДЕРЖАНИЕ

I. Лопушанский, *Несколько замечаний о теории электроно-фотонного каскада*

Из интегрального уравнения, аналогичного с уравнением Смолюховского в теории стохастических процессов, выведены известные дифференциально-интегральные уравнения для первых статистических моментов для электроно-фотонного каскада.

Кроме того представлены некоторые асимптоматичные свойства функции распределения для больших глубин абсорбера.

REFERENCES

Bhabha, H. J., *Proc. Roy. Soc.*, A **202**, 301 (1950).

Bhabha H. J., Chakrabarty S. K., *Proc. Roy. Soc.* **181**, 267 (1943); *Phys. Rev.*, **74**, 1352 (1948).

Jánossy L., Messel H., *Proc. Phys. Soc.* A **63**, 1101 (1950).

Konwent H., Łopuszański J., see this issue.

Landau L., Rumer G., *Proc. Roy. Soc.*, A **166**, 213 (1938).

Stachowiak H., see this issue:

Urbanik K., *Studia Mathematica* **15**, 34 (1955).

SOME PROPERTIES OF DISTRIBUTION FUNCTIONS OF ELECTRON — PHOTON CASCADES AT LARGE ABSORBER DEPTH

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It is shown that at large depth of absorber, the number of particles found in electron-photon cascades depends only on the spectrum of the primary photons,[1] and that at large depths, the number of particles decreases rapidly as the energy of the particles increases.

1. Introduction

The object of this paper is to investigate certain asymptotic relations between first moments and quantities connected with them which characterize the probability distribution for an electron-photon cascade at large absorber depth for threshold energies higher than the critical energy, since the work in this paper is done in approximation A. We have not made any assumptions as to the functional form of the cross sections except that we require the cross sections to be bounded functions which do not vanish within a finite interval in the range of their arguments. We further require that the functions will be homogeneous functions of the energy before and after collision (this condition is satisfied in the ultrarelativistic approximation with complete screening) and that the cross section for pair production will be smaller than the cross section for „Bremsstrahlung".

A similar treatment for nucleon cascades in a homogeneous nuclear matter was made by Urbanik (1955a) and Łopuszański (1955). The computational methods used in this paper are different from those used by these authors.

2. Notation

The electron-photon cascade is described by the function $P^{(i)}(n_1, n_2 | \varepsilon, x)$ where $= 1$ signifies that the primary particle was an electron and $i = 2$ a photon; $P^{(i)}(n_1, n_2 | \varepsilon, x)$ is the probability of finding n_1 electrons and n_2 photons of energy $> \varepsilon E_0$ in a cas-

[1] We assume, that the primary energy spectrum is monochromatic i.e. that all primary particles have the same energy.

cade at a depth x in the absorber, if the primary particle was of the type i, where E_0 is the energy of the primary particle.

If we are interested only in the number of photons, or only in the number of electrons, we introduce the function $P^{(i,k)}(n \mid \varepsilon, x)$ which represents the probability of finding n particles of type k.

We obtain the relation

$$P^{(i,k)}(n_k \mid \varepsilon, x) = \sum_{n_{3-k}=0}^{\infty} P^{(i)}(n_1, n_2 \mid \varepsilon, x) \tag{1}$$

The avarage number of particles is given by the formula

$$S^{(i,k)}(\varepsilon, x) = \sum_{n=0}^{\infty} n\, P^{(i,k)}(n \mid \varepsilon, x) \tag{2}$$

We introduce the generating function

$$G^{(i)}(\varepsilon, u_1, u_2, x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} P^{(i)}(n_1, n_2 \mid \varepsilon, x)\, u_1^{n_1} u_2^{n_2} \tag{3}$$

Function $G^{(i,k)}(\varepsilon, u, x)$ is defined as follows:

$$G^{(i,1)}(\varepsilon, u, x) = G^{(i)}(\varepsilon, u, 1, x) \sum_{n=0}^{\infty} P^{(i,1)}(n \mid \varepsilon, x)\, u^n \tag{4a}$$

$$G^{(i,2)}(\varepsilon, u, x) = G^{(i)}(\varepsilon, 1, u, x) = \sum_{n=0}^{\infty} P^{(i,2)}(n \mid \varepsilon, x)\, u^n \tag{4b}$$

We note that

$$G^{(i,k)}(\varepsilon, 0, x) = P^{(i,k)}(0, \varepsilon, x) \tag{5}$$

and therefore

$$P^{(i,k)}(>0 \mid \varepsilon, x) = 1 - G^{(i,k)}(\varepsilon, 0, x) \tag{6}$$

$P^{(i,k)}(>0 \mid \varepsilon, x)$ represents the probability of finding at least one particle of type k with an energy $> \varepsilon E_0$ at a depth of x. For simplicity we will write $P^{(i,k)}(>0 \mid \varepsilon, x)$ as $P^{(i,k)}(\varepsilon, x)$.

The Jánossy equation (Jánossy 1950) holds for the generating function in approximation A. Thus

$$\frac{\partial G^{(i)}(\varepsilon, u_1, u_2, x)}{\partial x} + a_i\, G^{(i)}(\varepsilon, u_1, u_2, x) =$$

$$= \int_0^1 G^{(1)}\left(\frac{\varepsilon}{1-\varepsilon'}, u_1, u_2, x\right) G^{(3-i)}\left(\frac{\varepsilon}{\varepsilon'}, u_1, u_2, x\right) w^{(i)}(\varepsilon')\, d\varepsilon' \tag{7}$$

where $w^{(i)}(\varepsilon')$ is the differential cross section for a particle of type i.

$$a_i = \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \tag{8}$$

3. Behaviour of the Mean Number of Particles for Large Absorber Depths

We will show that the following relations for the mean number of particles held to a first approximation:

$$\lim_{x\to\infty} \frac{S^{(i,k)}(\varepsilon', x)}{S^{(i,k)}(\varepsilon, x)} = 0 \text{ for } \varepsilon' > \varepsilon \tag{9}$$

$$\lim_{x\to\infty} \frac{S^{(1,k)}(\varepsilon, x)}{S^{(2,k)}(\varepsilon, x)} = 0 \tag{10}$$

$$\lim_{x\to\infty} \frac{S^{(2,k)}(\varepsilon', x)}{S^{(1,k)}(\varepsilon, x)} = 0 \qquad \varepsilon' > \varepsilon \tag{11}$$

$$\lim_{x\to\infty} \int_0^1 w^{(1)}(\varepsilon')\, d\varepsilon' \frac{S^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(1,k)}(\varepsilon, x)} = \alpha_1 - \alpha_2 \tag{12}$$

Proof:

As the starting point of our proof we will take the well-known equation for the first moments (Jánossy and Messel 1950)

$$\frac{\partial S^{(i,k)}(\varepsilon, x)}{\partial x} + \alpha_i S^{(i,k)}(\varepsilon, x) = \int_0^1 \left\{ S^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) + \right.$$

$$\left. + S^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) \right\} w^{(i)}(\varepsilon')\, d\varepsilon' \tag{13}$$

K. Urbanik (Urbanik 1955b) showed that the function $S^{(i,k)}(\varepsilon, x)$ satisfies the following relations

$$\lim_{x\to\infty} e^{\alpha_m x} S^{(i,k)}(\varepsilon, x) = \infty \qquad \alpha_m = \min(\alpha_1, \alpha_2)$$

$$\lim_{x\to\infty} e^{-\beta x} e^{\alpha_m x} S^{(i,k)}(\varepsilon, x) = 0 \qquad \beta > 0$$

As a result, $e^{\alpha_m x} S^{(i,k)}(\varepsilon, x)$ increases more slowly than an exponential function. We thus can write

$$S^{(i,k)}(\varepsilon, x) = e^{-\alpha_m x} W^{(i,k)}(\varepsilon, x) \tag{14}$$

where $W^{(i,k)}(\varepsilon, x)$ increases at a less-than-exponential rate.

Inserting equation (14) into the left-hand side of equation (13), we obtain

$$(\alpha_i - \alpha_m) e^{-\alpha_m x} W^{(i,k)}(\varepsilon, x) + e^{-\alpha_m x} \frac{\partial W^{(i,k)}(\varepsilon, x)}{\partial x} =$$

$$= \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \left[S^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) + S^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) \right] \tag{15}$$

From collision theory, we know that $a_1 > a_2$. Thus $a_m = a_2$. Dividing both sides by $S^{(i,\,k)}(\varepsilon, x)$, we obtain

$$(a_i - a_2) + \frac{1}{W^{(i,\,k)}(\varepsilon, x)} \frac{\partial W^{(i,\,k)}(\varepsilon, x)}{\partial x} =$$

$$= \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \left[\frac{S^{(1,\,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right)}{S^{(i,\,k)}(\varepsilon, x)} + \frac{S^{(3-i,\,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(i,\,k)}(\varepsilon, x)} \right] \tag{16}$$

We note that since $W^{(i,\,k)}(\varepsilon, x)$ does not increase as rapidly as an exponential function,

$$\lim_{x \to \infty} \frac{1}{W^{(i,\,k)}(\varepsilon, x)} \frac{\partial W^{(i,\,k)}(\varepsilon, x)}{\partial x} = 0$$

Therefore, as a result of (16), we obtain

$$\lim_{x \to \infty} \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \left[\frac{S^{(1,\,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right)}{S^{(1,\,k)}(\varepsilon, x)} + \frac{S^{(3-i,\,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(i,\,k)}(\varepsilon, x)} \right] = a_i - a_2 \tag{17}$$

Setting $i = 2$, and noting that $w^{(2)}(\varepsilon') = w^{(2)}(1-\varepsilon')$, we have

$$\lim_{x \to \infty} \int_0^1 w^{(2)}(\varepsilon')\, d\varepsilon' \frac{S^{(1,\,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(2,\,k)}(\varepsilon, x)} = 0 \tag{18}$$

Since $S^{(i,\,k)}(\varepsilon, x)$ is a non-negative function and $w^{(1)}(\varepsilon') > 0$ within the limits of integration

$$\lim_{x \to \infty} \frac{S^{(1,\,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(2,\,k)}(\varepsilon, x)} = 0 \tag{19}$$

In order to prove (10) a contrario we assume that $\lim_{x \to \infty} \frac{S^{(2,\,k)}(\varepsilon, x)}{S^{(1,\,k)}(\varepsilon, x)}$ is finite. Multiplying (19) by this expression, we obtain:

$$\lim_{x \to \infty} \frac{S^{(1,\,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(1,\,k)}(\varepsilon, x)} = 0$$

We multiply, in turn, this relation by $\lim_{x \to \infty} \frac{S^{(2,\,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(1,\,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}$ to obtain

$$\lim_{x \to \infty} \frac{S^{(2,\,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(1,\,k)}(\varepsilon, x)} = 0$$

For $i = 1$, equation (17) has the following form:

$$\lim_{x\to\infty} \int_0^1 w^{(1)}(\varepsilon')\, d\varepsilon' \left[\frac{S^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right)}{S^{(1,k)}(\varepsilon, x)} + \frac{S^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(1,k)}(\varepsilon, x)} \right] = a_1 - a_2 \qquad (20)$$

If we insert into (20) the two relations obtained previously, we obtain a result contrary to the assumption made in cascade theory that $w^{(i)}(\varepsilon')$ is bounded. Thus the assumption that $\lim_{x\to\infty} \frac{S^{(2,k)}(\varepsilon, x)}{S^{(1,k)}(\varepsilon, x)}$ is finite turns out to be incorrect. This ratio tends to infinity.

Therefore (10) is correct: $\lim_{x\to\infty} \frac{S^{(1,k)}(\varepsilon, x)}{S^{(2,k)}(\varepsilon, x)} = 0.$

Multiplying $\lim_{x\to\infty} \frac{S^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(1,k)}(\varepsilon, x)}$ by $\lim_{x\to\infty} \frac{S^{(1,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}$,

we obtain an expression analogous to the first term in the integrand of (20). This means that the first term in the integrand is small in comparison with the second, and therefore can be neglected. We thus obtain (12):

$$\lim_{x\to\infty} \int_0^1 w^{(1)}(\varepsilon')\, d\varepsilon' \frac{S^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(1,k)}(\varepsilon, x)} = a_1 - a_2$$

Also,

$$\lim_{x\to\infty} \int_0^1 w^{(1)}(\varepsilon')\, d\varepsilon' \frac{S^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right)}{S^{(1,k)}(\varepsilon, x)} = 0$$

From this last equation, we obtain

$$\lim_{x\to\infty} \frac{S^{(1,k)}(\varepsilon', x)}{S^{(1,k)}(\varepsilon, x)} = 0 \qquad \text{for } \varepsilon' > \varepsilon$$

which is the first part of (9). Proceeding in an analogous manner, we multiply (12) by $\lim_{x\to\infty} \frac{S^{(1,k)}(\varepsilon, x)}{S^{(2,k)}(\varepsilon, x)}$ and obtain the second part of (9):

$$\lim_{x\to\infty} \frac{S^{(2,k)}(\varepsilon', x)}{S^{(2,k)}(\varepsilon, x)} = 0 \qquad \text{for } \varepsilon' > \varepsilon$$

It now remains for us to prove Eq. (11). To do this, we will write Eq. (12) in the form

$$\lim_{x\to\infty}\left[\int_0^{\varepsilon''} w^{(1)}(\varepsilon')\,d\varepsilon'\,\frac{S^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'},x\right)}{S^{(1,k)}(\varepsilon,x)}+\int_{\varepsilon''}^{1} w^{(1)}(\varepsilon')\,d\varepsilon'\,\frac{S^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'},x\right)}{S^{(1,k)}(\varepsilon,x)}\right]=\alpha_1-\alpha_2,$$

where $(1-\varepsilon'')$ is a finite number. Because of (9), the first integral with the argument $\frac{\varepsilon}{\varepsilon'}>\varepsilon$ can be ignored in comparison with the second.

Thus

$$\lim_{x\to\infty}\int_0^{\varepsilon''} w^{(1)}(\varepsilon')\,d\varepsilon'\,\frac{S^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'},x\right)}{S^{(1,k)}(\varepsilon,x)}=0$$

as a result of which

$$\lim_{x\to\infty}\frac{S^{(2,k)}(\varepsilon',x)}{S^{(1,k)}(\varepsilon,x)}=0\quad\text{for }\varepsilon'>\varepsilon$$

4. *Extension of the Obtained Results on* $P^{(i,k)}(\varepsilon,x)$

Because of the relation (see the preceding paper, Łopuszański 1956)

$$\left[\frac{1}{\varepsilon}\right]^{-1}S^{(i,k)}(\varepsilon,x)\leqslant P^{(i,k)}(\varepsilon,x)\leqslant S^{(i,k)}(\varepsilon,x)\tag{21}$$

the ratio of $S^{(i,k)}(\varepsilon,x)$ to $P^{(i,k)}(\varepsilon,x)$ is a certain finite number greater than unity. Therefore, from Eqs. (9), (10), (11), and (12), we obtain

$$\lim_{x\to\infty}\frac{P^{(i,k)}(\varepsilon',x)}{P^{(i,k)}(\varepsilon,x)}=0\text{ for }\varepsilon'>\varepsilon\tag{22}$$

$$\lim_{x\to\infty}\frac{P^{(1,k)}(\varepsilon,x)}{P^{(2,k)}(\varepsilon,x)}=0\tag{23}$$

$$\lim_{x\to\infty}\frac{P^{(2,k)}(\varepsilon',x)}{P^{(1,k)}(\varepsilon,x)}=0\qquad\varepsilon'>\varepsilon\tag{24}$$

$$\lim_{x\to\infty}\int_0^{1} w^{(1)}(\varepsilon')\,d\varepsilon'\,\frac{P^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'},x\right)}{P^{(1,k)}(\varepsilon,x)}=C\tag{25}$$

where c is a finite number.

We obtain analogous results by putting (6) into the Jánossy equation (7):

$$\frac{\partial P^{(i,k)}(\varepsilon, x)}{\partial x} + a_i P^{(i,k)}(\varepsilon, x) = \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \left[P^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) + \right.$$

$$\left. + P^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) - P^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) \cdot P^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)\right] \tag{26}$$

Since from (14) and (21), $P^{(i,k)}(\varepsilon, x)$ decreases exponentially with the depth, we can, in the limit, neglect the last term in the integrand of (26) as being of small magnitude to a higher order. For large x, the equation

$$\frac{\partial P^{(i,k)}(\varepsilon, x)}{\partial x} + a_i P^{(i,k)}(\varepsilon, x) =$$

$$= \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \left[P^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) + P^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)\right] \tag{27}$$

is satisfied This equation is analogous to equation (13).
Because of (21), $P^{(i,k)}(\varepsilon, x)$ can be written in the form $P^{(i,k)}(\varepsilon, x) = e^{-a_m x} W^{(i,k)}(\varepsilon, x)$, and therefore, the same relations are satisfied for $P^{(i,k)}(\varepsilon, x)$ as for $S^{(i,k)}(\varepsilon, x)$. Thus (25) takes the form

$$\lim_{x\to\infty} \int_0^1 w^{(1)}(\varepsilon')\, d\varepsilon' \frac{P^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{P^{(1,k)}(\varepsilon, x)} = a_1 - a_2 . \tag{28}$$

5. *Some Properties of the Conditional Moments for Large Depths of Absorbers*

If $\overline{P}^{(i,k)}(n \mid \varepsilon, x)$ represents the probability of finding n particles, if there is at least one particle. Then the relation

$$P^{(i,k)}(n \mid \varepsilon, x) = \overline{P}^{(i,k)}(n \mid \varepsilon, x)\, P^{(i,k)}(>0 \mid \varepsilon, x) \text{ for } n > 0 \tag{29}$$

will be valid.

We define the conditional moment (the mean number of particles if there is at least one particle) to be $\overline{S}^{(i,k)}(\varepsilon, x) = \sum_{n=1}^{\infty} n \overline{P}^{(i,k)}(n \mid \varepsilon, x)$ (30)

Because of (2), (6), and (29)

$$\overline{S}^{(i,k)}(\varepsilon, x) = \frac{S^{(i,k)}(\varepsilon, x)}{P^{(i,k)}(\varepsilon, x)} \tag{31}$$

Evaluating $\frac{\partial \overline{S}^{(i,k)}(\varepsilon, x)}{\partial x}$ on the basis of Eqs. (31), (13), and (26), we obtain the following equation for $\overline{S}^{(i,k)}(\varepsilon, x)$

$$\frac{\partial \bar{S}^{(i,k)}(\varepsilon, x)}{\partial x} = \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \frac{P^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right)}{P^{(i,k)}(\varepsilon, x)} \left[\bar{S}^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) - \bar{S}^{(i,k)}(\varepsilon, x)\right] +$$

$$+ \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \frac{P^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{P^{(i,k)}(\varepsilon, x)} \left[\bar{S}^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) - \bar{S}^{(i,k)}(\varepsilon, x)\right] -$$

$$- \bar{S}^{(i,k)}(\varepsilon, x) \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \frac{P^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) P^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{P^{(i,k)}(\varepsilon, x)} \tag{32}$$

We assume that $S^{(i,k)}(\varepsilon, x)$ remains finite, then $\lim_{x\to\infty} \frac{\partial \bar{S}^{(i,k)}}{\partial x} = 0$. In the limit, we can ignore the last integral in (32). We then obtain

$$\lim_{x\to\infty} \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \frac{P^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right)}{P^{(i,k)}(\varepsilon, x)} \left[\bar{S}^{(1,k)}\left(\frac{\varepsilon}{1-\varepsilon'}, x\right) - \bar{S}^{(i,k)}(\varepsilon, x)\right] +$$

$$+ \lim_{x\to\infty} \int_0^1 w^{(i)}(\varepsilon')\, d\varepsilon' \frac{P^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{P^{(i,k)}(\varepsilon, x)} \left[\bar{S}^{(3-i,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) - \bar{S}^{(i,k)}(\varepsilon, x)\right] = 0 \tag{33}$$

Because of (22) and (23), we obtain an identity be letting $i = 2$. If we let $i = 1$, the first integral vanishes by (22), and there remains

$$\lim_{x\to\infty} \int_0^1 w^{(1)}(\varepsilon')\, d\varepsilon' \frac{P^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{P^{(1,k)}(\varepsilon, x)} \left[\bar{S}^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right) - \bar{S}^{(1,k)}(\varepsilon, x)\right] = 0 \tag{34}$$

Using the mean value theorem, we obtain, because of (28)

$$(\alpha_1 - \alpha_2) \lim_{x\to\infty} \left[\bar{S}^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'''(x)}, x\right) - \bar{S}^{(1,k)}(\varepsilon, x)\right] = 0 \tag{35}$$

Because of (24), we can use the small interval $(\varepsilon'', 1)$ for the limits of the integral of (34) instead of the limits from 0 to 1; $\varepsilon'''(x)$ will then lie near 1.

Since $S^{(i,k)}(\varepsilon, x)$ is continuous and bounded we can pass to the limits

$$\lim_{x\to\infty} [\bar{S}^{(2,k)}(\varepsilon, x) - \bar{S}^{(1,k)}(\varepsilon, x)] = 0 \tag{36}$$

Since $\overline{S}^{(i,k)}(\varepsilon, x)$ is always $\geqslant 1$, Eq. (36) signifies that in the limit

$$\overline{S}^{(1,k)}(\varepsilon, x) = \overline{S}^{(2,k)}(\varepsilon, x) \tag{37}$$

In the paper which follows (Konwent and Łopuszański 1956), it is shown that

$$\lim_{x \to \infty} \overline{S}^{(i,k)}(\varepsilon, x) = 1$$

6. *Physical Conclusions*

1) The spectrum of particles at large depths is very steep, i. e., if we somewhat raise the threshold energy, considerably fewer particles will be observed at large depths of the absorber.

2) If, at a large absorber depth, a particle is encountered, and if the primary particles are both photons and electrons, then the encountered particle is due to a photon; the electron-photon cascade does not display any ergodic properties. Only the primary photon spectrum influences the number of high energy particles counted at large depths (see footnote on page 181). We thus see that the „privileged" photons exist not only in the approximation taking into account ionization losses and the penetrating power of low-energy photons, but already exist in the ultra-relativistic approximation A.

The difficulties of experimental observation of this effect results from the fact that the ratio of the probability of encountering a particle due to a photon to the probability of encountering a particle due to an electron increases more slowly than an exponential function. while the probability of encountering a particle decreases exponentially. In order to check whether or not the effect has been observed, numerical calculations would have to be made. Thus the problem is to decide what is to be considered a large depth for a given threshold energy. At a depth where the above-mentioned ratio of probabilities is of such a size that the primary electrons can be neglected, the density of counted particles will be too small if the ratio increases too slowly. However an increase of this ratio with an increase in depth would confirm the results obtained in this paper.

The results we have obtained are weakened by the fact that the spectrum of particles at large depth is very steep, and if we somewhat raise the threshold energy of the particles derived from protons, and keep the energy of particles derived from electrons constant, the ratio of probabilities will decrease to zero with an increase in depth.

3) The functions $\lim_{x \to \infty} w^{(1)}(\varepsilon') \dfrac{P^{(2,k)}\left(> 0 \mid \frac{\varepsilon}{\varepsilon'}, x\right)}{P^{(1,k)}(> 0 \mid \varepsilon, x)}$

and $\lim_{x \to \infty} u^{(1)}(\varepsilon') \dfrac{S^{(2,k)}\left(\frac{\varepsilon}{\varepsilon'}, x\right)}{S^{(1,k)}(\varepsilon, x)}$ are of the Dirac δ type.

4) If, at large depths, we take into account only such cascades which have not as yet died out, then the mean number of observed particles will be the same regardless of wherther the primary particle was a photon or an electron.

Note: The term, „threshold energy" as used here should not be confused with the threshold of equipment since, in general, this is very low, while the theory employed here is valid only for high-energy particles.

I would like to thank Doctor J. Łopuszański for suggesting the topic, and for his valuable aid. I would also like to thank Professor J. Rzewuski, Professor M. Mięsowicz and his co-workers, and Professor A. Zawadzki for their helpful comments.

КРАТКОЕ СОДЕРЖАНИЕ

Г. Стаховяк, *О некоторых свойствах функции распределения электронно-фотонного каскада на большой глубине абсорбента.*

Доказано, что для электронно-фотонной составляющей космического излучения число частиц с энергией $>\varepsilon E_0$ встречающихся под толстым слоем абсорбента зависит единственно от первичного спектра фотонов[1], а также, что с возрастанием глубины абсорбента число регистрируемых частиц всё скорее уменьшаѐтся с возрастанием наименьшей энергии, которой должны обладать эти частицы, чтобы они были регистрированы (наименьшая энергия остаётся больше критической).

REFERENCES

Jánossy, L., *Proc. Phys. Soc.*, A **63**, 241 (1950).
Jánossy, L. and Messel, H., *Proc. Phys. Soc.* A **63**, 1101 (1950).
Konwent, H., and Łopuszański, J., see this issue.
Łopuszański, J., *Suppl. Nuovo Cimento* **2**, 1150, 1161 (1951); see this issue.
Urbanik, K., *Suppl. Nuovo Cimento* **2**, 1147 (1955a); Studia Math. **15**, 34 (1955b).

[1] Мы предполагаем, что енергии всех первичных частиц одинаковы.

SOME REMARKS ON THE ASYMPTOTIC BEHAVIOUR OF THE ELECTRON — PHOTON CASCADE FOR LARGE DEPTH OF THE ABSORBER

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(Received February 16, 1956)

It is shown that

$$\lim_{x\to\infty} \frac{S^{(i,2)}(\varepsilon', x)}{S^{(i,1)}(\varepsilon, x)} = 0 \qquad \text{for } \varepsilon' > \varepsilon$$

and

$$\lim_{x\to\infty} \frac{S^{(i,2)}(\varepsilon, x)}{S^{(i,1)}(\varepsilon, x)} = \infty$$

where $S^{(i,k)}(\varepsilon, x)$ denote the first moments for the electron-photon cascade, and further that the probability distribution function $P^{(i,k)}(n, m \mid \varepsilon, x)$ has the form

$$P^{(i)}(0,0 \mid \varepsilon, x) \sim 1$$
$$P^{(i)}(0,1 \mid \varepsilon, x) \sim S^{(i,2)}(\varepsilon, x)$$
$$P^{(i)}(n, m \mid \varepsilon, x) = o\left(P^{(i)}(0,1 \mid \varepsilon, x)\right)$$

for $(n, m) \neq (0,1)$ and $n + m > 0$.

1. *Introduction*

The present paper is a continuation of the two preceding papers (Łopuszański 1956, Stachowiak 1956, hereafter referred to as papers I and II resp.). We make use of the result obtained there. We also employ the Urbanik's conclusion (Urbanik 1955), that the first moments of the distribution may be written in form

$$S^{(i,k)}(\varepsilon, x) = e^{-\alpha^{(2)}x} W^{(i,k)}(\varepsilon, x) \qquad (1)$$

(for the notation used in this paper see paper I and II), where $W^{(i,k)}(\varepsilon, x)$ increases monotonically with increasing x and tends to infinity with increasing x not as rapidly as an exponential function. Further, we make use of an integral equation for the

generating function obtained from the formal solution of the G equations of Jànossy (1950) and given by Łopuszański (1955a). If we denote the generating function by $G(\varepsilon, u_1, u_2, x)$, this integral equation reads

$$\Gamma^{(i)}(\varepsilon, u_1, u_2, x) = (1-u_1)\, S^{(i,1)}(\varepsilon, x) + (1-u_2)\, S^{(i,2)}(\varepsilon, x) +$$

$$- \int_0^x d\xi \int_0^1 d\eta \int_0^1 \frac{d\varepsilon'}{\eta} s^{(i,1)}(\eta, x-\xi)\, w^{(1)}\left(\frac{\varepsilon'}{\eta}\right) \Gamma^{(2)}\left(\frac{\varepsilon}{\varepsilon'}, u_1, u_2, \xi\right) \Gamma^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, u_1, u_2, \xi\right) -$$

$$- \int_0^x d\xi \int_0^1 d\eta \int_0^1 \frac{d\varepsilon'}{\eta} s^{(i,2)}(\eta, x-\xi)\, w^{(2)}\left(\frac{\varepsilon'}{\eta}\right) \Gamma^{(1)}\left(\frac{\varepsilon}{\varepsilon'}, u_1, u_2, \xi\right) \Gamma^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, u_1, u_2, \xi\right) \tag{2}$$

where

$$\Gamma^{(i)}(\varepsilon, u_1, u_2, \xi) = 1 - G^{(i)}(\varepsilon, u_1, u_2, \xi)$$

We consider the asymptotic behaviour of the distribution function and first moments of the electron-photon cascade for large depth of absorber and positive threshold energy in approximation A.

From the mathematical point of view, our treatment is not quite rigorous, e. g. we assume the integration or differentiation operators to commute with the limit operator. Some indications as regards this question are given in the earlier papers of one of the authors (Łopuszański 1955), where analogous questions on the asymptotic behaviour of the nucleon cascade in homogeneous nuclear matter are considered.

We assume that the sections for bremsstrahlung and pair production, $w^{(1)}(E, E_0)$ and $w^{(2)}(E, E_0)$ respectively, are homogeneous functions of the energies of the particles before and after collision, that is,

$$w^{(i)}(E, E_0) = w^{(i)}\left(\frac{E}{E_0}\right)\frac{1}{E_0}$$

(in agreement with the well known expressions of Bethe and Heitler for the full screening approximation). We also assume that the above-mentioned cross-sections have no singularities and finaly that in the interval $0 \leqslant E \leqslant E_0$ no finite subintervals exist for which $w^{(i)}\left(\frac{E}{E_0}\right)$ is equal to zero. On the other hand no assumptions about the shape of the cross section functions are made.

We require, in accordance with field theory, only that

$$a^{(2)} = \int_0^1 w^{(2)}(\varepsilon)\, d\varepsilon < a^{(1)} = \int_0^1 w^{(1)}(\varepsilon)\, d\varepsilon$$

We emphasize, that from the inequality, given in paper I, for the probability $P^{(i,k)}(\varepsilon, x)$

$$P^{(i,k)}(\varepsilon, x) \leqslant S^{(i,k)}(\varepsilon, x) \leqslant \left[\frac{1}{\varepsilon}\right] P^{(i,k)}(\varepsilon, x) \tag{3}$$

it follows, that the quantities $P^{(i,k)}(\varepsilon, x)$ are of the same order of magnitude as $S^{(i,k)}(\varepsilon, x)$ respectively (if $P^{(i,k)}(\varepsilon, x)$ tends to infinity or to zero, then $S^{(i,k)}(\varepsilon, x)$ also tends to infinity or to zero, respectively). Consequently all relations obtained for $S^{(i,k)}(\varepsilon, x)$ also hold mutatis mutandis for $P^{(i,k)}(\varepsilon, x)$. As seen from paper I. we can also write

$$P^{(i,k)}(\varepsilon, x) = e^{-\alpha^{(2)} x} V^{(i,k)}(\varepsilon, x), \tag{4}$$

where $V^{(i,k)}(\varepsilon, x)$ tends to infinity as x approaches infinity more slowly than an exponential function.

Likewise we may easily generalize the result Urbanik to the functions

$$s^{(i,k)}(\varepsilon, x) = -\frac{\partial S^{(i,k)}(\varepsilon, x)}{\partial \varepsilon}$$

i. e. we may write

$$s^{(i,k)}(\varepsilon, x) = e^{-\alpha^{(2)} x} w^{(i,k)}(\varepsilon, x) \tag{5a}$$

where

$$\lim_{x \to \infty} w^{(i,k)}(\varepsilon, x) = \infty \tag{5b}$$

$$\lim_{x \to \infty} e^{-\beta x} w^{(i,k)}(\varepsilon, x) = 0 \qquad \beta > 0 \tag{5c}$$

We note also that almost all relations for first moments, deduced in paper II, are also valid for $s^{(i,k)}(\varepsilon, x)$ because the equations for $s^{(i,k)}(\varepsilon, x)$ are very similar to that fulfilled by $S^{(i,k)}(\varepsilon, x)$. In particular from

$$\lim_{x \to \infty} \frac{S^{(i,k)}(\varepsilon', x)}{S^{(i,k)}(\varepsilon, x)} = 0 \quad \varepsilon' > \varepsilon \tag{6}$$

it follows that

$$\lim_{x \to \infty} \frac{s^{(i,k)}(\varepsilon', x)}{s^{(i,k)}(\varepsilon, x)} = 0 \quad \varepsilon' > \varepsilon \tag{7}$$

2. Asymptotic behaviour of some quantities characterizing the distribution

The $s^{(i,k)}(\varepsilon, x)$ satisfy the following well known equations (see e. g. paper I)

$$\frac{\partial s^{(i,1)}(\varepsilon, x)}{\partial x} + \alpha^{(1)} s^{(i,1)}(\varepsilon, x) = \int_0^1 s^{(i,1)}(\varepsilon', x) w^{(1)}\left(1 - \frac{\varepsilon'}{\varepsilon}\right) \frac{d\varepsilon'}{\varepsilon'} + 2 \int_0^1 s^{(i,2)}(\varepsilon', x) w^{(2)}\left(\frac{\varepsilon}{\varepsilon'}\right) \frac{d\varepsilon'}{\varepsilon'} \tag{8}$$

$$\frac{\partial s^{(i,2)}(\varepsilon, x)}{\partial x} + \alpha^{(2)} s^{(i,2)}(\varepsilon, x) = \int_0^1 s^{(i,1)}(\varepsilon', x) w^{(1)}\left(\frac{\varepsilon}{\varepsilon'}\right) \frac{d\varepsilon'}{\varepsilon'} \tag{9}$$

Making use of (5a) we obtain from (8) and (9) respectively

$$\frac{\partial w^{(i,\,1)}(\varepsilon, x)}{\partial x} + (\alpha^{(1)} - \alpha^{(2)})\, w^{(i,\,1)}(\varepsilon, x) = \int_0^1 w^{(i,\,1)}(\varepsilon', x)\, w^{(1)}\left(1 - \frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} +$$

$$+ 2\int_0^1 w^{(i,\,2)}(\varepsilon', x)\, w^{(2)}\left(\frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} \tag{10}$$

$$\frac{\partial w^{(i,\,2)}(\varepsilon, x)}{\partial x} = \int_0^1 w^{(i,\,1)}(\varepsilon', x)\, w^{(1)}\left(\frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} \tag{11}$$

Let us investigate the behaviour of these equations for $x \to \infty$. For this purpose we divide both sides of (10) by $w^{(i,\,1)}(\varepsilon'', x)$, where $\varepsilon'' < \varepsilon$. We get

$$\frac{\dfrac{\partial w^{(i,\,1)}(\varepsilon, x)}{\partial x}}{w^{(i,\,1)}(\varepsilon'', x)} + (\alpha^{(1)} - \alpha^{(2)})\,\frac{w^{(i,\,1)}(\varepsilon, x)}{w^{(i,\,1)}(\varepsilon'', x)} = \int_0^1 \frac{w^{(i,\,1)}(\varepsilon', x)}{w^{(i,\,1)}(\varepsilon'', x)}\, w^{(1)}\left(1 - \frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} +$$

$$+ 2\int_0^1 \frac{w^{(i,\,2)}(\varepsilon', x)}{w^{(i,\,1)}(\varepsilon'', x)}\, w^{(2)}\left(\frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} \tag{12}$$

From (5b) and (5c) it follows that $w^{(ik)}(\varepsilon, x)$ increase with x more slowly than an exponential function. Consequently, we have

$$\lim_{x\to\infty} \frac{\dfrac{\partial w^{(i,\,k)}(\varepsilon, x)}{\partial x}}{w^{(i,\,k)}(\varepsilon, x)} = 0 \tag{13}$$

By (7) and (13) we have, obviously,

$$\lim_{x\to\infty} \frac{\dfrac{\partial w^{(i,\,k)}(\varepsilon, x)}{\partial x}}{w^{(i,\,k)}(\varepsilon'', x)} = 0 \quad \varepsilon > \varepsilon'' \tag{14}$$

Let x in (12) tend to infinity. By (7) and (14) we obtain from (12)

$$\lim_{x\to\infty} \int_0^1 \frac{w^{(i,\,2)}(\varepsilon', x)}{w^{(i,\,1)}(\varepsilon'', x)}\, w^{(2)}\left(\frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} = 0 \tag{15}$$

The other terms in (12) vanish. Since $w^{(2)}(\varepsilon)$ is positive and finite, we conclude from (15) that

$$\lim_{x\to\infty} \frac{w^{(i,\,2)}(\varepsilon', x)}{w^{(i,\,1)}(\varepsilon, x)} = \lim_{x\to\infty} \frac{s^{(i,\,2)}(\varepsilon', x)}{s^{(i,\,1)}(\varepsilon, x)} = 0 \qquad \varepsilon' > \varepsilon \tag{16}$$

Let us now divide both sides of (10) by $w^{(i,1)}(\varepsilon, x)$ and let x tend to infinity. Hence, by means of (7) and (13) we get

$$\lim_{x\to\infty} 2\int_0^1 \frac{w^{(i,2)}(\varepsilon', x)}{w^{(i,1)}(\varepsilon, x)}\, w^{(2)}\left(\frac{\varepsilon}{\varepsilon'}\right)\frac{d\varepsilon'}{\varepsilon'} = \alpha^{(1)} - \alpha^{(2)} \tag{17}$$

Making use of (16) and taking into account that the left-hand side of (17) does not vanish, it follows that

$$\lim_{x\to\infty} \frac{w^{(i,2)}(\varepsilon, x)}{w^{(i,1)}(\varepsilon, x)} = \lim_{x\to\infty} \frac{s^{(i,2)}(\varepsilon, x)}{s^{(i,1)}(\varepsilon, x)} = \infty \tag{18}$$

We note that relations (16) and (17), taken together, imply more than relations (16) and (18) taken together; from the former, we obtain some information about the type of singularity of the integrand in (17) (Dirac δ type singularity).

The relations (16) and (18) may be extended to $W^{(i,k)}(\varepsilon, x)$, namely

$$w^{(i,k)}(\varepsilon, x) = -\frac{\partial W^{(i,k)}(\varepsilon, x)}{\partial \varepsilon}$$

whence

$$\lim_{x\to\infty} \frac{W^{(i,2)}(\varepsilon', x)}{W^{(i,1)}(\varepsilon, x)} = \lim_{x\to\infty} \frac{-\int_{\varepsilon'}^1 \frac{w^{(i,2)}(\eta, x)}{w^{(i,2)}(\varepsilon', x)}\, d\eta}{\int_{\varepsilon}^1 \frac{w^{(i,1)}(\eta, x)}{w^{(i,2)}(\varepsilon', x)}\, d\eta} \tag{19}$$

where $\varepsilon' > \varepsilon$. From (7) and (16), it is seen that the numerator on the right-hand side of (19) vanishes and the denominator tends to infinity. Consequently

$$\lim_{x\to\infty} \frac{W^{(i,2)}(\varepsilon', x)}{W^{(i,1)}(\varepsilon, x)} = \lim_{x\to\infty} \frac{S^{(i,2)}(\varepsilon', x)}{S^{(i,1)}(\varepsilon, x)} = 0 \tag{20}$$

In similar manner we obtain

$$\lim_{x\to\infty} \frac{W^{(i,2)}(\varepsilon, x)}{W^{(i,1)}(\varepsilon, x)} = \lim_{x\to\infty} \frac{S^{(i,2)}(\varepsilon, x)}{S^{(i,1)}(\varepsilon, x)} = \infty \tag{21}$$

From (6) and (21) follows immediately, that

$$\lim_{x\to\infty} \frac{W^{(i,1)}(\varepsilon', x)}{W^{(i,2)}(\varepsilon, x)} = \lim_{x\to\infty} \frac{S^{(i,1)}(\varepsilon', x)}{S^{(i,2)}(\varepsilon, x)} = \lim_{x\to\infty} \frac{S^{(i,1)}(\varepsilon', x)}{S^{(i,1)}(\varepsilon, x)} \cdot \lim_{x\to\infty} \frac{S^{(i,1)}(\varepsilon, x)}{S^{(i,2)}(\varepsilon, x)} = 0 \tag{22}$$

On account of the inequality (3), it is easy to obtain from (20), (21), and (22) the following formulae for $P^{(i,k)}(\varepsilon, x)$

$$\lim_{x\to\infty} \frac{P^{(i,2)}(\varepsilon', x)}{P^{(i,1)}(\varepsilon, x)} = 0 \tag{23}$$

$$\lim_{x\to\infty} \frac{P^{(i,2)}(\varepsilon, x)}{P^{(i,1)}(\varepsilon, x)} = \infty \tag{24}$$

$$\lim_{x\to\infty} \frac{P^{(i,1)}(\varepsilon', x)}{P^{(i,2)}(\varepsilon, x)} = 0 \tag{25}$$

for $\varepsilon' > \varepsilon$.

A discussion of results (20) and (21) is given in the last section of this paper

3. *Asymptotic formulae for the distribution function*

Besides $P^{(i,k)}(\varepsilon, x)$, defined in paper I and II, we introduce $P^{(i)}(\varepsilon, x)$ the probability of finding at least one particle of energy larger than εE_0 at depth x due to a primary particle of type i and energy E_0.

We notice that we are able to define $P^{(i,k)}$ as well as $P^{(i)}$ in terms of the generating function $G(\varepsilon, u_1, u_2, x)$ (see Introduction of this paper), and so

$$\begin{aligned} P^{(i,k)}(\varepsilon, x) &= 1 - G^{(i)}(\varepsilon, u_1 = \delta_{2k}, u_2 = \delta_{1k}, x) = \\ &= 1^{(i)}(\varepsilon, u_1 = \delta_{2k}, u_2 = \delta_{1k}, x) \\ P^{(i)}(\varepsilon, x) &= 1^{(i)}(\varepsilon, O, O, x) \end{aligned} \tag{26}$$

In a manner similar to the case of $P^{(i,k)}(\varepsilon, x)$ (see paper I), we may show that

$$P^{(i)}(\varepsilon, x) = e^{-a^{(2)}x} V^{(i)}(\varepsilon, x) \tag{27}$$

where $V^{(i)}(\varepsilon, x)$ behaves in a manner similar to $V^{(i,k)}(\varepsilon, x)$ (see relation (4)). Consequently, using (2), we may at once write down the integral equations for $P^{(i)}(\varepsilon, x)$:

$$P^{(i)}(\varepsilon, x) = S^{(i,1)}(\varepsilon, x) + S^{(i,2)}(\varepsilon, x) - A_i P^{(i)}(\varepsilon, x) - B_i P^{(i)}(\varepsilon, x) \tag{28}$$

where

$$A_i P^{(i)}(\varepsilon, x) = \int_0^1 d\eta \int_0^1 \frac{d\varepsilon'}{\eta} w^{(1)}\left(\frac{\varepsilon'}{\eta}\right) \int_0^x s^{(i,1)}(\eta, x-\xi) P^{(2)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) P^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) d\xi \tag{29}$$

$$B_i P^{(i)}(\varepsilon, x) = \int_0^1 d\eta \int_0^1 \frac{d\varepsilon'}{\eta} w^{(2)}\left(\frac{\varepsilon'}{\eta}\right) \int_0^x s^{(i,2)}(\eta, x-\xi) P^{(1)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) P^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) d\xi$$

Let us divide both sides of (28) by $P^{(i)}(\varepsilon, x)$; we get

$$1 = \frac{S^{(i,1)}(\varepsilon, x) + S^{(i,2)}(\varepsilon, x)}{P^{(i)}(\varepsilon, x)} - A_i - B \tag{30}$$

We shall now show that

$$\lim_{x\to\infty} A_i = \lim_{x\to\infty} B_i = 0 \tag{31}$$

To prove (31) we write (29) in the form

$$A_i = C(\eta, \varepsilon') \int_0^x \frac{w^{(i,1)}(\eta, x-\xi)}{V^{(i)}(\varepsilon, x)} V^{(2)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) V^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) e^{-\alpha^{(2)}\xi}\, d\xi$$

$$B_i = D(\eta, \varepsilon') \int_0^x \frac{w^{(i,2)}(\eta, x-\xi)}{V^{(i)}(\varepsilon, x)} V^{(1)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) V^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) e^{-\alpha^{(2)}\xi}\, d\xi \qquad (32)$$

where we have used the notation (27) and (5a) and where

$$C(\eta, \varepsilon') = \int_0^1 d\eta \int_0^1 \frac{d\varepsilon'}{\eta} w^{(1)}\left(\frac{\varepsilon'}{\eta}\right)$$

$$D(\eta, \varepsilon') = \int_0^1 d\eta \int_0^1 \frac{d\varepsilon'}{\eta} w^{(2)}\left(\frac{\varepsilon'}{\eta}\right) \qquad (33)$$

Because of the properties of $w^{(i,k)}(\varepsilon, x)$ we may write

$$0 \leqslant A_i \leqslant C(\eta, \varepsilon') \frac{w^{(i,1)}(\eta, x)}{V^{(i)}(\varepsilon, x)} \int_0^x V^{(2)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) V^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) e^{-\alpha^{(2)}\xi}\, d\xi$$

$$0 \leqslant B_i \leqslant D(\eta, \varepsilon') \frac{w^{(i,2)}(\eta, x)}{V^{(i)}(\varepsilon, x)} \int_0^x V^{(1)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) V^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) e^{-\alpha^{(2)}\xi}\, d\xi \qquad (34)$$

We have

$$\lim_{x\to\infty} \int_0^x V^{(2)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) V^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) e^{-\alpha^{(2)}\xi}\, d\xi < \infty \qquad (35a)$$

$$\lim_{x\to\infty} \int_0^x V^{(1)}\left(\frac{\varepsilon'}{\varepsilon}, \xi\right) V^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) e^{-\alpha^{(2)}\xi}\, d\xi < \infty \qquad (35b)$$

We shall show only that (35a) is true, since (35b) may be proved in quite a similar manner. From (35a) we have

$$\int_0^\infty V^{(2)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) V^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) e^{-\alpha^{(2)}\xi}\, d\xi =$$

$$\int_0^\infty \left(V^{(2)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) e^{-\frac{\alpha^{(2)}\xi}{3}}\right)\left(V^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) e^{-\frac{\alpha^{(2)}\xi}{3}}\right) e^{-\frac{\alpha^{(2)}\xi}{3}}\, d\xi \leqslant$$

$$\leqslant \operatorname*{Max}_{0<\xi\leqslant\infty}\left(V^{(2)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) e^{-\frac{\alpha^{(2)}\xi}{3}}\right) \operatorname*{Max}_{0<\xi\leqslant\infty}\left(V^{(1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) e^{-\frac{\alpha^{(2)}\xi}{3}}\right) \int_0^\infty e^{-\frac{\alpha^{(2)}\xi}{3}}\, d\xi < \infty$$

in regard to properties of $V^{(i)}(\varepsilon, x)$. This proves (35a). Now we shall show that

$$\lim_{x\to\infty} \frac{w^{(i,1)}(\eta, x)}{V^{(i)}(\varepsilon, x)} = 0 \tag{36a}$$

$$\lim_{x\to\infty} \frac{w^{(i,2)}(\eta, x)}{V^{(1)}(\varepsilon, x)} = 0 \tag{36b}$$

To do this it suffices to show that

$$\lim_{x\to\infty} \frac{W^{(i,1)}(\eta, x)}{V^{(i)}(\varepsilon, x)} = 0 \qquad \eta > \varepsilon \tag{37a}$$

and

$$\lim_{x\to\infty} \frac{W^{(i,2)}(\eta, x)}{V^{(i)}(\varepsilon, x)} = 0 \qquad \eta > \varepsilon \tag{37b}$$

since $w^{(i,k)}(\varepsilon, x)$ is always non-negative.
From (29) we see that for all η, used there, we have $\eta > 2\varepsilon$, thus, of course, $\eta > \varepsilon$. It is easily to show that

$$P^{(i)}(\varepsilon, x) \geqslant P^{(i,2)}(\varepsilon, x) \tag{38a}$$

and

$$P^{(i)}(\varepsilon, x) \geqslant P^{(i,1)}(\varepsilon, x) \tag{38b}$$

Thus, on account of (38), we have

$$0 \leqslant \frac{S^{(i,1)}(\eta, x)}{P^{(i)}(\varepsilon, x)} \leqslant \frac{S^{(i,1)}(\eta, x)}{P^{(i,1)}(\varepsilon, x)} \tag{39a}$$

and

$$0 \leqslant \frac{S^{(i,2)}(\eta, x)}{P^{(i)}(\varepsilon, x)} \leqslant \frac{S^{(i,2)}(\eta, x)}{P^{(i,2)}(\varepsilon, x)} \tag{39b}$$

Futher, on the basis of (3) and (6) we have

$$\lim_{x\to\infty} \frac{S^{(i,1)}(\eta, x)}{P^{(i,1)}(\varepsilon, x)} \leqslant \lim_{x\to\infty} \left[\frac{1}{\varepsilon}\right] \frac{S^{(i,1)}(\eta, x)}{S^{(i,1)}(\varepsilon, x)} = 0 \tag{40a}$$

and

$$\lim_{x\to\infty} \frac{S^{(i,2)}(\eta, x)}{P^{(i,2)}(\varepsilon, x)} \leqslant \lim_{x\to\infty} \left[\frac{1}{\varepsilon}\right] \frac{S^{(i,2)}(\eta, x)}{S^{(i.2)}(\varepsilon, x)} = 0 \tag{40b}$$

Relations (39) and (40) complete the proof of (37), and consequently of (36) too.

Now we are ready to complete the proof of (31). Thus, in agreement with (35), the integrals in (34) are positive and bounded for $x \to \infty$ and, since (36) holds, we have the results stated.

Consequently we are able to write

$$\lim_{x\to\infty} \frac{S^{(i,1)}(\varepsilon, x) + S^{(i,2)}(\varepsilon, x)}{P^{(i)}(\varepsilon, x)} = 1 \tag{41}$$

Let us write (41) in the form

$$\lim_{x\to\infty} \frac{S^{(i,2)}(\varepsilon, x)}{P^{(i)}(\varepsilon, x)} \left\{1 + \frac{S^{(i,1)}(\varepsilon, x)}{S^{(i,2)}(\varepsilon, x)}\right\} = 1$$

Then, because of (21), we have finally

$$\lim_{x\to\infty} \frac{S^{(i,2)}(\varepsilon, x)}{P^{(i)}(\varepsilon, x)} = 1 \tag{42}$$

Formulae (41) and (42) taken together enable us to obtain the asymptotic formulae for the distribution function $P^{(i)}(n, m|\varepsilon, x)$ (see paper II).
We have

$$\begin{aligned} &P^{(i)}(0,0|\varepsilon, x) \sim 1 \\ &P^{(i)}(0,1|\varepsilon, x) \sim S^{(i,2)}(\varepsilon, x) \\ &P^{(i)}(n, m|\varepsilon, x) = o\,(P^{(i)}(0,1|\varepsilon, x) \end{aligned} \tag{43}$$

for pairs of numbers (n, m), different from (0,1), where $n + m > 0$.

In a very similar manner we obtain from (2), with help of (26), the integral equations for $P^{(i,k)}(\varepsilon, x)$

$$1 = \frac{S^{(i,1)}(\varepsilon, x)}{P^{(i,1)}(\varepsilon, x)} - A_{i1} - B_{i1} \tag{44a}$$

$$1 = \frac{S^{(i,2)}(\varepsilon, x)}{P^{(i,2)}(\varepsilon, x)} - A_{i2} - B_{i2} \tag{44b}$$

where

$$\begin{aligned} A_{i1} &= C(\eta, \varepsilon') \int_0^x \frac{S^{(i,1)}(\eta, x-\xi)}{P^{(i,1)}(\varepsilon, x)} P^{(2,1)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) P^{(1,1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) d\xi \\ B_{i1} &= D(\eta, \varepsilon') \int_0^x \frac{S^{(i,2)}(\eta, x-\xi)}{P^{(i,1)}(\varepsilon, x)} P^{(1,1)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) P^{(1,1)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) d\xi \\ A_{i2} &= C(\eta, \varepsilon') \int_0^x \frac{S^{(i,1)}(\eta, x-\xi)}{P^{(i,2)}(\varepsilon, x)} P^{(2,2)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) P^{(1,2)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) d\xi \\ B_{i2} &= D(\eta, \varepsilon') \int_0^x \frac{S^{(i,2)}(\eta, x-\xi)}{P^{(i,2)}(\varepsilon, x)} P^{(1,2)}\left(\frac{\varepsilon}{\varepsilon'}, \xi\right) P^{(1,2)}\left(\frac{\varepsilon}{\eta-\varepsilon'}, \xi\right) d\xi \end{aligned} \tag{45}$$

We now show, that

$$\lim_{x\to\infty} A_{i1} = \lim_{x\to\infty} B_{i1} = \lim_{x\to\infty} A_{i2} = \lim_{x\to\infty} B_{i2} = 0 \tag{46}$$

The proof of (46) is very similar to that given above for (31). The only thing which needs some explanation is the proof of the relation

$$\lim_{x\to\infty} \frac{w^{(i,\,k)}(\eta, x)}{V^{(i,\,l)}(\varepsilon, x)} = 0$$

for $\eta > \varepsilon$ and $i, k, l, = 1,2$.
From (6), (20) and (22), using a method similar to that used in case of (36), we obtain, for $\eta > \varepsilon$,

$$\lim_{x\to\infty} \frac{w^{(i,\,1)}(\eta, x)}{V^{(i,\,1)}(\varepsilon, x)} = 0 \tag{47a}$$

$$\lim_{x\to\infty} \frac{w^{(i,\,2)}(\eta, x)}{V^{(i,\,2)}(\varepsilon, x)} = 0 \tag{47b}$$

$$\lim_{x\to\infty} \frac{w^{(i,\,2)}(\eta, x)}{V^{(i,\,1)}(\varepsilon, x)} = 0 \tag{47c}$$

$$\lim_{x\to\infty} \frac{w^{(i,\,1)}(\eta, x)}{V^{(i,\,2)}(\varepsilon, x)} = 0 \tag{47b}$$

Thus, since (46) holds we obtain finally from (44)

$$\lim_{x\to\infty} \frac{S^{(i,\,1)}(\varepsilon, x)}{P^{(i,\,1)}(\varepsilon, x)} = 1 \tag{48a}$$

$$\lim_{x\to\infty} \frac{S^{(i,\,2)}(\varepsilon, x)}{P^{(i,\,2)}(\varepsilon, x)} = 1 \tag{48b}$$

4. *Conclusions*

We now recapitulate the main results of the present paper and try to give some physical interpretation to them. We will also consider these results in light of the results obtained in paper II.
The relations

$$\lim_{x\to\infty} \frac{S^{(i,\,2)}(\varepsilon, x)}{S^{(i,\,1)}(\varepsilon, x)} = \infty \tag{21}$$

and

$$\lim_{x\to\infty} \frac{S^{(i,\,2)}(\varepsilon', x)}{S^{(i,\,1)}(\varepsilon, x)} = 0 \quad \text{for } \varepsilon' > \varepsilon \tag{20}$$

form together our first main result. Equation (21) may be interpreted as follows:
The ratio of the average number of photons to the average number of electrons

tends to infinity if the depth of the absorber increases to infinity independently of the type of primary particle initiating the cascade, provided the threshold energy of the apparatus registrating electrons and photons is the same. In other words, if the depth of the absorber is sufficiently large, the number of registrated photons is greater than the number of registrated electrons and this ratio will increase with increasing depth. Now relation (20) will „weaken" the „strong" relation (21). We interpret (21) as follows:

The ratio of the average number of photons of energy larger than $\varepsilon' E_0$ to the average number of electrons of energy larger than εE_0, where $\varepsilon' > \varepsilon$, tends to zero if the depth of absorber tends to infinity independently of the type of primary particle. In other words: If the depth of the absorber is sufficiently large and the threshold energy for the photons is higher than for the electrons, then the number of registrated electrons is greater than the number of registrated photons and this ratio increases with increasing depth.

As was said in the Introduction, our computations are done in the approximation A, i. e. for threshold energies chosen so that neither Greisen's photon effect (Greisen 1949) nor the ionization has any significance. It is well known that these two factors act in a direction to stregthen the influence of photons and suppress the influence of the electrons (both, however, of low energy) at great depth, i. e., their action is in agreement with (21). But our result — as was already said — is independent of the two effects mentioned above. Therefore it seems to us that it would be interesting to verify our results experimentally for particles of higher energy, where Greisen's photon effect and the ionization do not obliterate our effect. We are, however, aware of some experimental difficulties. Although (21) holds, nevertheless for large but finite, depths, we do not expect the effect to be strong, because $\alpha^{(2)}$ baryly differs from $\alpha^{(1)}$. One may, however, compare the ratio of the numbers of photons and electrons for various depths. A further difficulty is that the number of photons as well as the number of electrons, falls off very rapidly with increasing energy for large depths. As already mentioned, this steepness of the average-number curve, considerably weakens the effect implied by (21). Consequently, one may expect that experiment would indicate that the effects of (20) and (21) blend into each other.

Our results, as well as the results of paper II, are in agreement with the numerical data and curves for the ratio of first moments of the photon-electron cascade given by Jánossy and Messel (1951) and obtained by means of the saddle — point method. The derivatives of the functions represented by these curves are always positive. These function behave like $e^{a\sqrt{x}}$ where $a > 0$.

The second main result is that

$$\begin{aligned} &P^{(i)}(0,0|\varepsilon, x) \sim 1 \\ &P^{(i)}(0,1|\varepsilon, x) \sim S^{(i,2)}(\varepsilon, x) \\ &P^{(i)}(n, m|\varepsilon, x) = o\,(P^{(i)}(0,1|\varepsilon, x)), \end{aligned} \tag{43}$$

where $n + m > 0$ and $(n, m) \neq (0,1)$, or, written in terms of conditional probalities

$$\bar{P}^{(i)}(0,1|\varepsilon, x) = \frac{P^{(i)}(0,1|\varepsilon, x)}{\sum\limits_{n+m>0}\sum P^{(i)}(n, m|\varepsilon, x)} \sim 1$$

$$\bar{P}^{(i)}(n, m \mid \varepsilon, x) = \frac{P^{(i)}(n, m \mid \varepsilon, x)}{\sum\limits_{n+m>0}\sum P^{(i)}(n, m|\varepsilon, x)} \sim 0 \tag{43a}$$

We interpret (43a) as follows:

if the depth of the absorber is sufficiently large and if we have registrated at least one particle, we may be sure that only one particle was registrated and that this particle was a photon.

This statement means that the predominant part of registration events form the single particle registrations; other registrations may be neglected at large depth.

In our considerations no attention was paid to the kind of the primary particle, initiating the cascade. In paper II, however, the role played by the primary particle was investigated exhaustively. Considering our results in light of the results contained in paper II we are able to conclude as follows:

if the depth of the absorber is sufficiently large and if we encounter at least one particle, we may be sure that there is only one particle. This particle is a photon belonging to a cascade due to a primary photon.

Formulae (43) give the asymptotic solution of the distribution function for x tending to infinity. Therefore (43) is the first approximation for the distribution function for very large depth. In order to find the distribution function for a thinner absorber one is expected to find further approximations. These approximations are probably built up from higher factorial moments of the distribution.

Eq. (48) may be interpreted in (41) and (42), which lead to (43). Thus: if at large depth, we have registrated at least one electron (photon) and an arbitrary number of photons (electrons), we may be sure that we have registrated only one electron (photon) and an arbitrary number of photons (electrons).

Finally we emphasize that in course of our work we did not make any assumptions about the shape of the cross-section functions, except for the few general assumtions noted in the Introduction.

Our results are independent of the manner in which the particles in the cascade multiply. The only essential feature is that $\alpha^{(2)} < \alpha^{(1)}$.

We wish to express our thanks to Professor J. Rzewuski, to Cracow group of cosmic ray physicists especially to Professor M. Mięsowicz as well as to Professor A. Zawadzki of the University of Łódź, and to Mr. H. Stachowiak for helpful discussions.

КРАТКОЕ СОДЕРЖАНИЕ

Конвент и Лопушанский, *Несколько замечаний о асимптоматических свойствах электроно- фотоновой каскады для больших глубин абсорбера*

В работе показано, что:

$$\lim_{x\to\infty} \frac{S^{(i,2)}(\varepsilon', x)}{S^{(i,1)}(\varepsilon, x)} = 0 \qquad \text{для } \varepsilon' > \varepsilon$$

а также:

$$\lim_{x\to\infty} \frac{S^{(i,2)}(\varepsilon, x)}{S^{(i,1)}(\varepsilon, x)} = \infty$$

где: $S^{(i\,k)}(\varepsilon, x)$ обозначают первые статистические моменты для электроно-фотонного каскада, притом, что функция распределения вероятности

$$P^{(i)}(0,0 \mid \varepsilon, x) \sim 1$$

$$P^{(i)}(0,1 \mid \varepsilon, x) \sim S^{(i,2)}(\varepsilon, x)$$

$$P^{(i)}(n, m \mid \varepsilon, x) = o\,(P^{(i)}(0,1 \mid \varepsilon, x)$$

$$(n, m) \neq (0,1);\ n + m > 0.$$

REFERENCES

Greisen, K., *Phys. Rev.*, **75**, 1071 (1949).
Jánossy, L., *Proc. Phys. Soc.*, **A 64**, 802 (1950).
Jánossy, L. and Messel, H., *Proc. Roy. Irish. Acad.*, **A 54**, 217 (1951).
Łopuszański, J., *Acta Phys. Polon.*, **14**, 191 (1955a).
Łopuszański, J., *Suppl. Nouvo Cimento*, **4**, 1150 (1955b); 4, 1161 (1955c).
Łopuszański, J., (1956), see this issue.
Stachowiak, H., (1956), see this issue.
Urbanik, K., *Studia Mathematica* **15**, 34 (1955).

Volumen XV (1956) — Fasciculus 3